

A Numerical Scheme for BSVIEs*

Yanqing Wang[†]

May 17, 2016

Abstract

In this paper, we consider the Euler method for backward stochastic Volterra integral equations. First, we approximate the original equation by a family of backward stochastic equations (BSDEs, for short). Then we solve the BSDEs by the Euler method. Finally, by virtue of the numerical solutions to BSDEs, we get the numerical solution to original equation and obtain the global $1/2$ order convergence speed in L^2 norm.

Keywords: Backward stochastic Volterra integral equation, the Euler method, backward stochastic differential equation, Malliavin analysis.

AMS subject classification: 60H20, 65C30

1 Introduction

Throughout this paper, we let $T \in (0, +\infty)$, $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete probability space and $\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}$ be the natural filtration generalized by a 1-dimensional Wiener process $\{W(t) : t \in [0, T]\}$ satisfying the usual conditions. The purpose of this work is to present a numerical scheme for solving the following backward stochastic Volterra integral equation (BSVIE, for short):

$$(1.1) \quad Y(t) = g(t, x(T)) + \int_t^T f(t, s, x(s), Y(s), Z(t, s))ds - \int_t^T Z(t, s)dW(s), \quad t \in [0, T],$$

where $f : \Delta^c \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^n$ are given maps with $\Delta^c = \{(t, s) \in [0, T]^2 : t < s\}$, and $x(\cdot)$ satisfies the following stochastic Volterra integral equation (SVIE, for short):

$$(1.2) \quad x(t) = \varphi(t) + \int_0^t b(t, s, x(s))ds + \int_0^t \sigma(t, s, x(s))dW(s), \quad t \in [0, T].$$

Here $\varphi : [0, T] \times \Omega \rightarrow \mathbb{R}^d$, $b, \sigma : [0, T]^2 \times \mathbb{R}^d \rightarrow \mathbb{R}^d$.

BSVIEs are natural and nontrivial extensions of backward stochastic differential equations (BSDEs, for short), and the general BSVIEs can not be reduced to BSDEs (see [17]). The main feature of SVIEs/BSVIEs is that these equations contain memories, which is closer to reality. We refer

*This work is supported in part by the National Natural Science Foundation of China (11526167), the Fundamental Research Funds for the Central Universities (SWU113038, XDJK2014C076), the Natural Science Foundation of CQCSTC (2015jcyjA00017).

[†]School of Mathematics and Statistics, Southwest University, Chongqing 400715, China; email: yqwang@amss.ac.cn

to [3], [10] for the pioneering work on SVIEs. Nonlinear BSVIEs was first introduced in 2002 ([11]). Later, Yong ([17]) studied the well-posedness of solutions to generalized BSVIEs. Thereafter, BSVIEs turned out to be an extremely useful tool in the study of stochastic control problems for SVIEs, time-inconsistent stochastic differential utility and risk management (see, e.g., [6, 16]).

Generally, it is impossible to obtain the true solutions to BSDEs/BSVIEs. Hence, the study of numerical solutions becomes necessary and interesting. In recent period, the study of numerical solutions to stochastic differential equations (SDEs, for short) becomes an active topic. So far, the following numerical schemes for BSDEs have been presented: the four step scheme, the Euler method, the random walk approach, the Wiener chaos expansion method, the finite transposition method and so on (see, e.g., [1, 4, 5, 7, 12, 13, 14, 19]). But for BSVIE, the numerical method is quiet limited. Here we mention [2]. In [2], the numerical method for the following BSVIE is considered:

$$(1.3) \quad Y(t) = g(t, W) + \int_t^T f(s, Y(s))ds - \int_t^T Z(t, s)dW(s), \quad t \in [0, T],$$

which is approximated by a family of discrete BSVIEs driven by a binary random walk with solutions $(Y^{(n)}, Z^{(n)})$. Under suitable conditions, $Y^{(n)}$ converges weakly to Y in the Skorokhod topology. That result relies on a representation for BSVIEs by systems of quasilinear PDEs of parabolic type.

In this paper, we employ the Euler method to present the numerical solution to BSVIE (1.1). To be specific, suppose a partition $\pi : 0 = t_0 < t_1 < \dots < t_N = T$ of $[0, T]$ with the mesh size $|\pi| = \max_{0 \leq i \leq N} |t_{i+1} - t_i|$. Then we denote $\Delta_i = t_{i+1} - t_i$ and $\Delta_i W = W(t_{i+1}) - W(t_i)$, for $i = 0, 1, \dots, N-1$.

For $0 \leq k \leq N-1$, we present the Euler method for BSVIE (1.1) as follows:

$$(1.4) \quad \begin{cases} Y^{k,\pi}(t_N) = g(t_k, x^\pi(T)), \\ Y^{k,\pi}(t_l) = \mathbb{E}\left(Y^{k,\pi}(t_{l+1}) + f(t_k, t_l, x^\pi(t_l), Y^{l,\pi}(t_{l+1}), Z^{k,\pi}(t_l))\Delta_l \middle| \mathcal{F}_{t_l}\right), \\ Z^{k,\pi}(t_l) = \mathbb{E}\left(\frac{\Delta_l W}{\Delta_l}(Y^{k,\pi}(t_{l+1}) + f(t_k, t_l, x^\pi(t_l), Y^{l,\pi}(t_{l+1}), Z^{k,\pi}(t_l))) \middle| \mathcal{F}_{t_l}\right), \\ k \leq l \leq N-1. \end{cases}$$

Here $x^\pi(\cdot)$ is the numerical solution to SVIE (1.2) stated as

$$(1.5) \quad \begin{cases} x^\pi(0) = x^\pi(t_0) = \varphi(0), \\ x^\pi(t_{i+1}) = \varphi(t_{i+1}) + \sum_{k=0}^i \left[b(t_{i+1}, t_k, x^\pi(t_k))\Delta_k + \sigma(t_{i+1}, t_k, x^\pi(t_k))\Delta_k W \right], \\ i = 0, 1, \dots, N-1. \end{cases}$$

Under suitable conditions on f, g, φ, b and σ (assumptions (A1)–(A4) below), in the cases: (I) $f = f(t, s, x, y)$; (II) $f = f(t, s, x, z)$, we can prove that (Theorem 4.3)

$$(1.6) \quad \max_{0 \leq k \leq N} \mathbb{E}|Y(t_k) - Y^{k,\pi}(t_k)|^2 + \sum_{k=0}^{N-1} \mathbb{E} \int_{t_k}^{t_{k+1}} \int_t^T |Z(t, s) - Z^{k,\pi}(\tau(s))|^2 ds \leq K|\pi|,$$

where $\tau(\cdot)$ is a map on $[0, T)$ defined by $\tau(s) = t_i, s \in [t_i, t_{i+1}), i = 0, 1, \dots, N-1$ and K is a constant.

The rest of the paper is organized as follows: In Section 2, we review some of the standard results on SDEs and BSDEs, introduce our general setting and show the well-posedness of SVIE (1.2) and BSVIE (1.1). In Section 3, we present the Euler method to obtain the numerical solution to SVIE (1.2) and get the convergence speed. In Section 4, we adopt the Euler method for BSVIE (1.1), and the convergence and error analysis are also provided. A numerical example is presented in Section 5.

2 Preliminaries

Recall that \mathbb{R}^n is the n -dimensional Euclidean space with the standard Euclidean norm $|\cdot|$ induced by the standard Euclidean inner product $\langle \cdot, \cdot \rangle$. Hereafter, the superscript $^\top$ denotes the transpose of a vector or a matrix. We now introduce some spaces: for $p, q \geq 1$,

- $L_{\mathcal{F}_T}^p(\Omega; \mathbb{R}^n)$ is the space of all \mathcal{F}_T -measurable random variances ξ valued in \mathbb{R}^n such that

$$\|\xi\|_{L_{\mathcal{F}_T}^p(\Omega; \mathbb{R}^n)} = (\mathbb{E}|\xi|^p)^{\frac{1}{p}} < \infty.$$

- $L_{\mathbb{F}}^p(\Omega; L^q(0, T; \mathbb{R}^n))$ is the space of all \mathbb{F} -progressively measurable processes $\varphi(\cdot)$ valued in \mathbb{R}^n such that

$$\|\varphi(\cdot)\|_{L_{\mathbb{F}}^p(\Omega; L^q(0, T; H))} = \left[\mathbb{E} \left(\int_0^T |\varphi(t)|^q dt \right)^{\frac{p}{q}} \right]^{\frac{1}{p}} < \infty.$$

When $p = q$, we write $L_{\mathbb{F}}^p(\Omega \times (0, T); \mathbb{R}^n)$ for simplicity.

- $\mathbb{L}_a^{1,2}(\mathbb{R}^n)$ is the space of all \mathbb{F} -progressively measurable processes $u(\cdot)$ valued in \mathbb{R}^n satisfying

- (i) For almost all $t \in [0, T]$, $u(t) \in \mathbb{D}^{1,2}(\mathbb{R}^n)$;
- (ii) $\mathbb{E} \left(\int_0^T |u(t)|^2 dt + \int_0^T \int_0^T |D_\theta u(t)|^2 d\theta dt \right) < \infty$.

The following lemma collects some standard results in SDE and BSDE literature. We only list them.

Lemma 2.1. *Suppose that $b_0, \sigma_0 : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $f_0 : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are \mathbb{F} -adapted random fields, satisfying:*

- (a) *they are uniformly Lipschitz continuous with respect to $x \in \mathbb{R}^d, y \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$,*
- (b) *$b_0(\cdot, 0), \sigma_0(\cdot, 0) \in L_{\mathbb{F}}^2(\Omega \times (0, T); \mathbb{R}^d), f_0(\cdot, 0, 0) \in L_{\mathbb{F}}^2(\Omega \times (0, T); \mathbb{R}^n)$.*

For any $x \in \mathbb{R}^d$ and $\xi \in L_{\mathcal{F}_T}^2(\Omega; \mathbb{R}^n)$, $X(\cdot)$ is the solution to the following SDE:

$$X(t) = x + \int_0^t b_0(s, X(s)) ds + \int_0^t \sigma_0(s, X(s)) dW(s), \quad t \in [0, T],$$

and $(Y(\cdot), Z(\cdot))$ solves the BSDE:

$$Y(t) = \xi + \int_t^T f_0(s, Y(s), Z(s)) ds - \int_t^T Z(s) dW(s), \quad t \in [0, T].$$

Then, for any $p \geq 2$, we have the following estimates:

$$\begin{aligned} \mathbb{E}\left(\sup_{0 \leq t \leq T} |X(t)|^p\right) &\leq C \left\{ |x|^p + \mathbb{E}\left(\int_0^T |b_0(t, 0)| dt\right)^p + \mathbb{E}\left(\int_0^T |\sigma_0(t, 0)|^2 dt\right)^{p/2} \right\}, \\ \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y(t)|^p\right) + \mathbb{E}\left(\int_0^T |Z(s)|^2 ds\right)^{p/2} &\leq C \left\{ \mathbb{E}|\xi|^p + \mathbb{E}\left(\int_0^T |f_0(s, 0, 0)| ds\right)^p \right\}, \end{aligned}$$

where C is a constant.

Throughout the paper, we will make use of the following assumptions.

(A1) $f : [0, T]^2 \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, and there exists a constant L such that

$$\begin{aligned} (2.1) \quad &|f(t_1, s_1, x, y, z) - f(t_2, s_2, x, y, z)| \leq L(|t_1 - t_2|^{1/2} + |s_1 - s_2|^{1/2}), \\ &s_1, s_2 \in (\max\{t_1, t_2\}, T], \quad x \in \mathbb{R}^d, \quad y, z \in \mathbb{R}^n, \\ &|f(\cdot, \cdot, 0, 0, 0)| \leq L, \end{aligned}$$

and f has continuous and uniformly bounded first and second partial derivatives with respect to x, y and z (boundary is L).

(A2) $g : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^n$, and there exists a constant L such that

$$\begin{aligned} (2.2) \quad &|g(t_1, x) - g(t_2, x)| \leq L|t_1 - t_2|^{1/2}, \quad t_1, t_2 \in [0, T], \quad x \in \mathbb{R}^d, \\ &|g(\cdot, 0)| \leq L, \end{aligned}$$

and g has continuous and uniformly bounded first and second partial derivatives with respect to x (boundary is L).

(A3) $b, \sigma : [0, T]^2 \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, and there exists a constant L such that

$$\begin{aligned} (2.3) \quad &|b(t_1, s_1, x) - b(t_2, s_2, x)| + |\sigma(t_1, s_1, x) - \sigma(t_2, s_2, x)| \leq L(|t_1 - t_2|^{1/2} + |s_1 - s_2|^{1/2}), \\ &t_1, t_2, s_1, s_2 \in [0, T], \quad x \in \mathbb{R}^d, \\ &|b(\cdot, \cdot, 0)| + |\sigma(\cdot, \cdot, 0)| \leq L, \end{aligned}$$

and b, σ has continuous and uniformly bounded first and second partial derivatives with respect to x (boundary is L).

(A4) $\varphi(\cdot)$ is \mathbb{F} -adapted continuous process and there exists a constant $p_0 > 2$ and L such that

$$\begin{aligned} (2.4) \quad &\mathbb{E}|\varphi(t) - \varphi(s)|^2 \leq L|t - s|, \quad t, s \in [0, T], \\ &\mathbb{E}|D_{\theta_1}\varphi(t) - D_{\theta_2}\varphi(t)|^2 \leq L|\theta_1 - \theta_2|, \quad 0 \leq \theta_1, \theta_2 \leq t \leq T, \\ &\sup_{0 \leq \theta_1, \theta_2 \leq t \leq T} \mathbb{E}\left[|\varphi(t)|^{2p_0} + |D_{\theta_1}\varphi(t)|^{2p_0} + |D_{\theta_1}D_{\theta_2}\varphi(t)|^{p_0}\right] \leq L^{2p_0}. \end{aligned}$$

In what follows, K and C are positive constants, depending only on L and T , and may be different from line to line.

2.1 Regularity of $x(\cdot)$

In this part, we review the wellposedness of SVIE (1.2). Under assumptions (A3)–(A4), the wellposedness of SVIEs can be proved by a routine successive approximation argument ([10]). The following properties on $x(\cdot)$ are need later.

Lemma 2.2. Under assumptions (A3)–(A4), for any $0 \leq t_0 \leq t \leq T$, $0 \leq \theta_1, \theta_2 \leq t$, it holds that

$$(2.5) \quad \begin{aligned} \sup_{0 \leq \theta_1, \theta_2 \leq t \leq T} \mathbb{E} \left(|x(t)|^{2p_0} + |D_{\theta_1} x(t)|^{2p_0} + |D_{\theta_1} D_{\theta_2} x(t)|^{p_0} \right) &< K, \\ \mathbb{E} |x(t) - x(t_0)|^2 &\leq K(t - t_0), \\ \sup_{\theta_1, \theta_2 \leq t \leq T} \mathbb{E} |D_{\theta_1} x(t) - D_{\theta_2} x(t)|^2 &\leq K|\theta_1 - \theta_2|, \end{aligned}$$

where K is a constant depending only on p_0 , L and T .

Proof. Suppose that $0 \leq t_0 \leq t \leq T$. Then by SVIE (1.2), one obtains

$$(2.6) \quad \begin{aligned} \mathbb{E} |x(t)|^{2p_0} &\leq 3^{2p_0-1} \mathbb{E} |\varphi(t)|^{2p_0} + (3T)^{2p_0-1} \mathbb{E} \int_0^t |b(t, s, x(s))|^{2p_0} ds \\ &\quad + 3^{2p_0-1} T^{\frac{2p_0}{2}-1} \mathbb{E} \int_0^t |\sigma(t, s, x(s))|^{2p_0} ds \\ &\leq 3^{2p_0-1} L^{2p_0} + (6T)^{2p_0-1} \mathbb{E} \int_0^t |b(t, s, x(s)) - b(t, s, 0)|^{2p_0} + |b(t, s, 0)|^{2p_0} ds \\ &\quad + 6^{2p_0-1} T^{\frac{2p_0}{2}-1} \mathbb{E} \int_0^t |\sigma(t, s, x(s)) - \sigma(t, s, 0)|^{2p_0} + |\sigma(t, s, 0)|^{2p_0} ds \\ &\leq K + K \mathbb{E} \int_0^t |x(s)|^{2p_0} ds. \end{aligned}$$

Consequently, by virtue of Gronwall's inequality, we can get $\sup_{0 \leq t \leq T} \mathbb{E} |x(t)|^{2p_0} < K$. Also by the routine successive approximation argument, $D_\theta x(\cdot)$, the Malliavin derivative of $x(\cdot)$, satisfies the following SVIE: for any $0 \leq \theta \leq t \leq T$

$$D_\theta x(t) = D_\theta \varphi(t) + \sigma(t, \theta, x(\theta)) + \int_\theta^t b_x(t, s, x(s)) D_\theta x(s) ds + \int_\theta^t \sigma_x(t, s, x(s)) D_\theta x(s) dW(s).$$

Similarly, we also can obtain $\sup_{0 \leq t \leq T, 0 \leq \theta_1, \theta_2 \leq t} \mathbb{E} \left(|D_{\theta_1} x(t)|^{2p_0} + |D_{\theta_1} D_{\theta_2} x(t)|^{p_0} \right) < K$, which is the first inequality of (2.5).

Now, making use of the first inequality of (2.5), with the similar estimate to that of (2.6), we can obtain

$$\begin{aligned} \mathbb{E} |x(t) - x(t_0)|^2 &\leq 5 \mathbb{E} |\varphi(t) - \varphi(t_0)|^2 + 5T \mathbb{E} \int_0^{t_0} |b(t, s, x(s)) - b(t_0, s, x(s))|^2 ds \\ &\quad + 5 \mathbb{E} \int_0^{t_0} |\sigma(t, s, x(s)) - \sigma(t_0, s, x(s))|^2 ds \\ &\quad + 5 \mathbb{E} \left| \int_{t_0}^t b(t, s, x(s)) ds \right|^2 + 5 \mathbb{E} \int_{t_0}^t |\sigma(t, s, x(s))|^2 ds \\ &\leq K|t - t_0| + K \mathbb{E} \int_{t_0}^t |x(s)|^2 ds \\ &\leq K|t - t_0|, \end{aligned}$$

which is the second inequality of (2.5).

For the third inequality of (2.5), suppose that $\theta_2 \leq \theta_1 \leq t$. Since

$$\begin{aligned} D_{\theta_1}x(t) - D_{\theta_2}x(t) &= (D_{\theta_1}\varphi(t) - D_{\theta_2}\varphi(t)) + (\sigma(t, \theta_1, x(\theta_1)) - \sigma(t, \theta_2, x(\theta_2))) \\ &\quad + \int_{\theta_1}^t b_x(D_{\theta_1}x(t) - D_{\theta_2}x(t))ds + \int_{\theta_1}^t \sigma_x(D_{\theta_1}x(t) - D_{\theta_2}x(t))dW(s) \\ &\quad - \int_{\theta_2}^{\theta_1} b_x D_{\theta_2}x(t)ds - \int_{\theta_2}^{\theta_1} \sigma_x D_{\theta_2}x(t)dW(s), \end{aligned}$$

it easy to calculate that

$$\begin{aligned} &\mathbb{E}|D_{\theta_1}x(t) - D_{\theta_2}x(t)|^2 \\ &= 6\mathbb{E}|D_{\theta_1}\varphi(t) - D_{\theta_2}\varphi(t)|^2 + 6L^2(|\theta_1 - \theta_2| + \mathbb{E}|x(\theta_1) - x(\theta_2)|^2) \\ &\quad + 6(T+1)L^2\mathbb{E}\int_{\theta_1}^t |D_{\theta_1}x(s) - D_{\theta_2}x(s)|^2ds + 6(T+1)L^2\mathbb{E}\int_{\theta_2}^{\theta_1} |D_{\theta_2}x(s)|^2ds \\ &\leq K|\theta_2 - \theta_1| + 6(T+1)L^2\mathbb{E}\int_{\theta_1}^t |D_{\theta_1}x(s) - D_{\theta_2}x(s)|^2ds. \end{aligned}$$

Hence, by Gronwall's inequality, we have

$$\sup_{\theta_1, \theta_2 \leq t \leq T} \mathbb{E}|D_{\theta_1}x(t) - D_{\theta_2}x(t)|^2 \leq K|\theta_1 - \theta_2|,$$

completing the proof. ■

2.2 Regularity of $(Y(\cdot), Z(\cdot, \cdot))$

The following result on wellposedness of BSVIE (1.1) comes from [17, Theorem 3.7 and 4.1].

Theorem 2.3. *Under assumptions (A1)–(A4), BSVIE (1.1) admits a unique solution $(Y(\cdot), Z(\cdot, \cdot))$. Moreover, the following estimates holde:*

$$\begin{aligned} (2.7) \quad &\mathbb{E}\int_S^T |Y(t)|^2dt + \mathbb{E}\int_S^T \int_t^T |Z(t, s)|^2dsdt \\ &\leq C\left\{\mathbb{E}\int_S^T |g(t, x(T))|^2dt + \mathbb{E}\int_S^T \left(\int_t^T |f(t, s, 0, 0, 0)|ds\right)^2dt\right\}, \text{ for any } S \in [0, T], \end{aligned}$$

$$\begin{aligned} (2.8) \quad &\sum_{i=1}^n \mathbb{E}\left\{\int_S^T |D_r^i Y(t)|^2dt + \int_S^T \int_t^T |D_r^i Z(t, s)|^2dsdt\right\} \\ &\leq C\mathbb{E}\left\{\int_S^T |g(t, x(T))|^2dt + \sum_{i=1}^n \int_S^T |D_r^i g(t, x(T))|^2dt\right. \\ &\quad \left. + \int_S^T \left(\int_t^T |f(t, s, 0, 0, 0)|ds\right)^2dt\right\}, \text{ for any } r, S \in [0, T]. \end{aligned}$$

Morevoer, $(D_r^i Y(\cdot), D_r^i Z(\cdot, \cdot))$ is the adapted solution to the following BSVIE:

$$\begin{aligned} (2.9) \quad &D_r^i Y(t) = D_r^i g(t, x(T)) + \int_t^T \left(g_x(t, s, x(s), Y(s), Z(t, s))D_r^i x(s)\right. \\ &\quad + g_y(t, s, x(s), Y(s), Z(t, s))D_r^i Y(s) \\ &\quad + g_z(t, s, x(s), Y(s), Z(t, s))D_r^i Z(t, s)\Big)ds \\ &\quad - \int_r^T D_r^i Z(t, s)dW(s), \quad t \in [r, T]. \end{aligned}$$

In addition, for any $0 \leq t < u \leq T$, $1 \leq i \leq n$,

$$\begin{aligned}
(2.10) \quad Z_i(t, u) = & D_u^i g(t, x(T)) + \int_u^T \left(f_x(t, s, x(s), Y(s), Z(t, s)) D_u^i x(s) \right. \\
& + f_y(t, s, x(s), Y(s), Z(t, s)) D_u^i Y(s) \\
& + f_z(t, s, x(s), Y(s), Z(t, s)) D_u^i Z(t, s) \Big) ds \\
& - \int_u^T D_u^i Z(t, s) dW(s).
\end{aligned}$$

The following result is used to deduce the convergence speed in the Euler method for BSVIE (1.1).

Lemma 2.4. *Under assumptions (A1)–(A4), for any $t, t_0 \in [0, T]$, it holds that*

$$(2.11) \quad \mathbb{E}|Y(t) - Y(t_0)|^2 + \mathbb{E} \int_{t \vee t_0}^T |Z(t, s) - Z(t_0, s)|^2 ds \leq C|t - t_0|,$$

where C is a constant.

Proof. Suppose that $t_0 < t$. By [17, Corrolary 3.6], under assumptions (A1)–(A4), we have

$$\begin{aligned}
(2.12) \quad & \mathbb{E}|Y(t) - Y(t_0)|^2 + \mathbb{E} \int_t^T |Z(t, s) - Z(t_0, s)|^2 ds \\
& \leq C \left\{ \mathbb{E}|g(t, x(T)) - g(t_0, x(T))|^2 + \mathbb{E} \left(\int_{t_0}^t |f(t_0, s, x(s), Y(s), Z(t_0, s))| ds \right)^2 \right. \\
& \quad + \mathbb{E} \left(\int_t^T |f(t, s, x(s), Y(s), Z(t, s)) - f(t_0, s, x(s), Y(s), Z(t, s))| ds \right)^2 \\
& \quad \left. + \mathbb{E} \int_{t_0}^t |Z(t_0, s)|^2 ds \right\} \\
& \leq C|t - t_0| + C \mathbb{E} \int_{t_0}^t (|Y(s)|^2 + |Z(t_0, s)|^2) ds.
\end{aligned}$$

For $\mathbb{E}|Y(\cdot)|^2$, also by [17, Corrolary 3.6], one has

$$\begin{aligned}
\mathbb{E}|Y(t)|^2 + \mathbb{E} \int_t^T |Z(t, s)|^2 ds & \leq C \left\{ \mathbb{E}|g(t, x(T))|^2 + \mathbb{E} \left(\int_t^T |f(t, s, x(s), Y(s), 0)| ds \right)^2 \right\} \\
& \leq C + C \mathbb{E} \int_t^T |Y(s)|^2 ds.
\end{aligned}$$

By Gronwall's inequality, one gets that $\sup_{0 \leq t \leq T} \mathbb{E}|Y(t)|^2 \leq C$. Thus

$$(2.13) \quad \mathbb{E} \int_{t_0}^t |Y(s)|^2 ds \leq C|t - t_0|.$$

Setting $t = t_0$ in (2.10), by [17, Corollary 3.6], Lemma 2.2 and (2.8), we can obtain

$$\begin{aligned}
& \mathbb{E}|Z(t_0, u)|^2 + \mathbb{E} \int_u^T |D_u Z(t_0, s)|^2 ds \\
& \leq C \left\{ \mathbb{E}|D_u g(t_0, x(T))|^2 + \mathbb{E} \left(\int_u^T (|f_x(t_0, s, x(s), Y(s), Z(t_0, x)) D_u x(s)| \right. \right. \\
(2.14) \quad & \quad \left. \left. + |f_y(t_0, s, x(s), Y(s), Z(t_0, x)) D_u Y(s)|) ds \right)^2 \right\} \\
& \leq C \left\{ \mathbb{E}|D_u x(T)|^2 + \mathbb{E} \int_u^T (|D_u x(s)|^2 + |D_u Y(s)|^2) ds \right\} \\
& < \infty.
\end{aligned}$$

Now, (2.12), together with (2.13) and (2.14), yields that

$$\mathbb{E}|Y(t) - Y(t_0)|^2 + \mathbb{E} \int_t^T |Z(t, s) - Z(t_0, s)|^2 ds \leq C|t - t_0|,$$

which is (2.11). ■

3 The Euler method for SVIEs

The aim of this section is to review the Euler method for SVIE (1.2) under assumptions (A3)–(A4). For numerical solutions to general SVIEs with singular kernels, one can refer to [20].

For simplicity, throughout this paper, we assume that $\Delta_i = |\pi| = \frac{T}{N} \leq 1$, for each $i = 0, 1, \dots, N-1$. Our numerical scheme still works for general uniform partition of $[0, T]$ (i.e., there exists a constant K , such that $K|\pi| \leq \Delta_j$, for any $j = 0, 1, \dots, N-1$). We also need the following two functions $\tau(\cdot)$ and $\pi(\cdot)$ defined on $[0, T]$ by

$$(3.1) \quad \tau(t) = t_i, \quad \pi(t) = i, \quad t \in [t_i, t_{i+1}), \quad i = 0, 1, \dots, N-1.$$

The Euler method for SVIE (1.2) is as follows:

$$(3.2) \quad \begin{cases} x^\pi(0) = x^\pi(t_0) = \varphi(0), \\ x^\pi(t_{i+1}) = \varphi(t_{i+1}) + \sum_{k=0}^i \left(b(t_{i+1}, t_k, x^\pi(t_k)) \Delta_k + \sigma(t_{i+1}, t_k, x^\pi(t_k)) \Delta_k W \right), \\ i = 0, 1, \dots, N-1. \end{cases}$$

In order to obtain the convergent speed, we introduce the following SVIE related to (3.2):

$$(3.3) \quad x^\pi(t) = \varphi(t) + \int_0^t b(t, \tau(s), x^\pi(\tau(s))) ds + \int_0^t \sigma(t, \tau(s), x^\pi(\tau(s))) dW(s), \quad t \in [0, T].$$

Now, we are in the step to obtain the convergent speed for the Euler method (3.2). By SVIEs (1.2) and (3.3), one has, for any $t \in [0, T]$,

$$\begin{aligned}
x(t) - x^\pi(t) &= \int_0^t (b(t, s, x(s)) - b(t, \tau(s), x^\pi(\tau(s)))) ds \\
&\quad + \int_0^t (\sigma(t, s, x(s)) - \sigma(t, \tau(s), x^\pi(\tau(s)))) dW(s).
\end{aligned}$$

A direct calculation leads to

$$\begin{aligned}
& \mathbb{E}|x(t) - x^\pi(t)|^2 \\
& \leq 2T\mathbb{E} \int_0^t |b(t, s, x(s)) - b(t, \tau(s), x^\pi(\tau(s)))|^2 ds \\
(3.4) \quad & + 2\mathbb{E} \int_0^t |\sigma(t, s, x(s)) - \sigma(t, \tau(s), x^\pi(\tau(s)))|^2 ds \\
& \leq 2(T+1)L^2\mathbb{E} \int_0^t \left((s - \tau(s)) + |x(s) - x^\pi(s)|^2 + |x^\pi(s) - x^\pi(\tau(s))|^2 \right) ds.
\end{aligned}$$

By the definition of τ in (3.1) (suppose that $t \in [t_i, t_{i+1})$, $i = 0, 1, \dots, N-1$),

$$(3.5) \quad \mathbb{E} \int_0^t (s - \tau(s)) ds = \sum_{k=0}^{i-1} \int_{t_k}^{t_{k+1}} (s - t_k) ds + \int_{t_i}^t (s - t_i) ds = \sum_{k=0}^{i-1} \frac{\Delta_k^2}{2} + \frac{(t - t_i)^2}{2} \leq T|\pi|.$$

Now, supposing that $t \in [t_i, t_{i+1})$, we estimate $\mathbb{E}|x^\pi(t) - x^\pi(\tau(t))|^2$. By SVIE (3.3), one has

$$\begin{aligned}
& x^\pi(t) - x^\pi(\tau(t)) = x^\pi(t) - x^\pi(t_i) \\
& = (\varphi(t) - \varphi(t_i)) + \int_0^{t_i} \left(b(t, \tau(s), x^\pi(\tau(s))) - b(t_i, \tau(s), x^\pi(\tau(s))) \right) ds \\
& + \int_0^{t_i} \left(\sigma(t, \tau(s), x^\pi(\tau(s))) - \sigma(t_i, \tau(s), x^\pi(\tau(s))) \right) dW(s) \\
& + \int_{t_i}^t b(t, t_i, x^\pi(t_i)) ds + \int_{t_i}^t \sigma(t, t_i, x^\pi(t_i)) dW(s).
\end{aligned}$$

Then, under assumptions (A3)–(A4), it is easy to check that

$$\begin{aligned}
(3.6) \quad \mathbb{E}|x^\pi(t) - x^\pi(\tau(t))|^2 & \leq 5\mathbb{E}|\varphi(t) - \varphi(t_i)|^2 + 5 \left(\int_0^{t_i} L\sqrt{t - t_i} ds \right)^2 + 5 \int_0^{t_i} L^2(t - t_i) ds \\
& + 5T\mathbb{E} \int_{t_i}^{t_{i+1}} |b(t, t_i, x^\pi(t_i))|^2 ds + 5\mathbb{E} \int_{t_i}^{t_{i+1}} |\sigma(t, t_i, x^\pi(t_i))|^2 ds \\
& \leq C|\pi| + C\mathbb{E}|x^\pi(t_i)|^2|\pi|.
\end{aligned}$$

For $\mathbb{E}|x^\pi(t_i)|^2$, also by SVIE (3.3),

$$\begin{aligned}
\mathbb{E}|x^\pi(t)|^2 & \leq 3\mathbb{E}|\varphi(t)|^2 + 6T\mathbb{E} \int_0^t |b(t, \pi(s), 0)|^2 ds + 6TL^2 \int_0^t |x^\pi(\pi(s))|^2 ds \\
& + 6\mathbb{E} \int_0^t |\sigma(t, \pi(s), 0)|^2 ds + 6L^2 \int_0^t |x^\pi(\pi(s))|^2 ds \\
& \leq C + 6L^2(T+1) \int_0^t \mathbb{E}|x^\pi(\pi(s))|^2 ds.
\end{aligned}$$

Setting $g(t) = \sup_{s \in [0, t]} \mathbb{E}|x^\pi(s)|^2$, by Gronwall's inequality, one obtains

$$(3.7) \quad g(t) \leq Ce^{6L^2T(T+1)}, \text{ for all } t \in [0, T].$$

(3.6), together with (3.7), yields that

$$(3.8) \quad \mathbb{E}|x^\pi(t) - x^\pi(\tau(t))|^2 \leq C|\pi|, \forall t \in [0, T].$$

By (3.4), (3.5) and (3.8), we have

$$\mathbb{E}|x(t) - x^\pi(t)|^2 \leq C|\pi| + 2L^2(T+1) \int_0^t \mathbb{E}|x(s) - x^\pi(s)|^2 ds,$$

which, by Gronwall's inequality, deduces that

$$\sup_{t \in [0, T]} \mathbb{E}|x(t) - x^\pi(t)|^2 \leq e^{2L^2(T+1)T} C|\pi|.$$

By the above analysis, we get the following convergence speed of the Euler method (3.2) for SVIE (1.2).

Theorem 3.1. *Let (A3)–(A4) hold. Then for $x(\cdot)$ and $x^\pi(\cdot)$ defined as in (1.2) and (3.2), respectively, there exists a constant C , depending only on L and T , such that*

$$(3.9) \quad \max_{0 \leq i \leq N} \mathbb{E}|x(t_i) - x^\pi(t_i)|^2 \leq C|\pi|.$$

4 The Euler method for BSVIEs

In this section, we mainly present the Euler method to calculate the numerical solution to BSVIE (1.1), and prove the convergence speed of that method for (1.1). For $1 \leq k \leq N-1$, we present the Euler method for BSVIE (1.1) as follows:

$$(4.1) \quad \begin{cases} Y^{k,\pi}(t_N) = g(t_k, x^\pi(T)), \\ Y^{k,\pi}(t_l) = \mathbb{E}\left(Y^{k,\pi}(t_{l+1}) + f(t_k, t_l, x^\pi(t_l), Y^{l,\pi}(t_{l+1}), Z^{k,\pi}(t_l))\Delta_l \middle| \mathcal{F}_{t_l}\right), \\ Z^{k,\pi}(t_l) = \mathbb{E}\left(\frac{\Delta_l W}{\Delta_l}(Y^{k,\pi}(t_{l+1}) + f(t_k, t_l, x^\pi(t_l), Y^{l,\pi}(t_{l+1}), Z^{k,\pi}(t_l))\Delta_l) \middle| \mathcal{F}_{t_l}\right), \\ k \leq l \leq N-1. \end{cases}$$

Here $x^\pi(\cdot)$ ($k \leq l \leq N-1$) is defined by (3.2).

In order to obtain the convergence speed, we introduce $(Y^k(\cdot), Z^k(\cdot))$ ($k = 0, 1, \dots, N-1$) solving the following BSDE:

$$(4.2) \quad \begin{cases} dY^k(s) = -f(t_k, s, x(s), Y^l(s), Z^k(s))ds + Z^k(s)dW(s), \quad s \in (t_l, t_{l+1}], \quad k+1 \leq l \leq N-1, \\ dY^k(s) = -f(t_k, s, x(s), Y^k(s), Z^k(s))ds + Z^k(s)dW(s), \quad s \in [t_k, t_{k+1}], \\ Y^k(T) = g(t_k, x(T)), \quad Y^k(t_l) = Y^k(t_l + 0), \quad k+1 \leq l \leq N-1, \end{cases}$$

and $(Y^{k,\pi}(\cdot), \widehat{Z}^{k,\pi}(\cdot))$ ($k = 0, 1, \dots, N-1$) solving the following BSDE:

$$(4.3) \quad \begin{cases} Y^{k,\pi}(t_{l+1}) - Y^{k,\pi}(t) = -f(t_k, t_l, x^\pi(t_l), Y^{l,\pi}(t_{l+1}), \widehat{Z}_0^{k,\pi}(t_l))\Delta_l + \int_t^{t_{l+1}} \widehat{Z}^{k,\pi}(s)dW(s), \\ t \in (t_l, t_{l+1}], \quad k+1 \leq l \leq N-1, \\ Y^{k,\pi}(t_{k+1}) - Y^{k,\pi}(t) = -f(t_k, t_k, x^\pi(t_k), Y^{k,\pi}(t_{k+1}), \widehat{Z}_0^{k,\pi}(t_k))\Delta_k + \int_t^{t_{k+1}} \widehat{Z}^{k,\pi}(s)dW(s), \\ t \in [t_k, t_{k+1}], \\ Y^{k,\pi}(T) = g(t_k, x^\pi(T)), \quad Y^{k,\pi}(t_l) = Y^{k,\pi}(t_l + 0), \quad k+1 \leq l \leq N-1, \\ \widehat{Z}_0^{k,\pi}(t_N) = 0, \quad \widehat{Z}_0^{k,\pi}(t_l) = \frac{1}{\Delta_l} \mathbb{E}\left(\int_{t_l}^{t_{l+1}} \widehat{Z}^{k,\pi}(u)du \middle| \mathcal{F}_{t_l}\right), \quad k \leq l \leq N-1. \end{cases}$$

Remark 4.1. (i) When $|\pi| < L^2$, BSDE (4.3) admits a unique solution.

(ii) By (4.1) and BSDE (4.3), in the cases: (I) $f = f(t, s, x, y)$; (II) $f = f(t, s, x, z)$, we can easily check that, for any $k = 0, 1, \dots, N-1$, $k+1 \leq j \leq N-1$,

$$(4.4) \quad Z^{k,\pi}(t_j) = \frac{1}{\Delta_j} \mathbb{E} \left(\int_{t_j}^{t_{j+1}} \widehat{Z}^{k,\pi}(u) du | \mathcal{F}_{t_j} \right) = \widehat{Z}_0^{k,\pi}(t_j).$$

By the definition of $\tau(\cdot)$ and $\pi(\cdot)$ in (3.1), we can define $Y^{\pi(t)}(t) = Y^k(t)$, $Z^{\pi(t)}(s) = Z^k(s)$, and $Y^{\pi(t),\pi}(\tau(t)) = Y^{k,\pi}(t_k)$, $Z^{\pi(t),\pi}(\tau(s)) = Z^{k,\pi}(t_j)$, for $t \in [t_k, t_{k+1})$, $k = 0, 1, \dots, N-1$, and $s \geq t$, $s \in [t_j, t_{j+1}]$, $k \leq j \leq N-1$. The following result comes from [18].

Theorem 4.2. Let (A1)–(A4) hold. Then, BSDE (4.2) admits a unique solution $(Y^{\pi(\cdot)}(\cdot), Z^{\pi(\cdot)}(\cdot))$, and

$$\mathbb{E} \int_0^T |Y(t) - Y^{\pi(t)}(t)|^2 dt + \mathbb{E} \int_0^T \int_t^T |Z(t, s) - Z^{\pi(t)}(s)|^2 ds dt \leq K|\pi|,$$

where K is a constant only depending on L and T .

Now we state our main result on convergence speed of the Euler method (4.1) for BSVIE (1.1).

Theorem 4.3. Suppose that $f = f(t, s, x, y)$ or $f = f(t, s, x, z)$ in BSDE (4.2), and let (A1)–(A4) hold. Then

$$(4.5) \quad \sup_{0 \leq t \leq T} \mathbb{E} |Y(\tau(t)) - Y^{\pi(t),\pi}(\tau(t))|^2 + \mathbb{E} \int_0^T \int_t^T |Z(t, s) - Z^{\pi(t),\pi}(\tau(s))|^2 ds \leq K|\pi|,$$

where K is a constant depending only on L and T .

The proof of Theorem 4.3 is lengthy, we split it into several lemmas.

4.1 Regularity of $(Y^{\pi(\cdot)}(\cdot), Z^{\pi(\cdot)}(\cdot))$

In this part, we mainly study the regularity of $Y^{\pi(\cdot)}(\cdot)$ and $Z^{\pi(\cdot)}(\cdot)$, which is crucial in proving Theorem 4.3. First, we need the following lemma.

Lemma 4.4. (1) Suppose that $a_i \geq 0$, $b_i \geq 0$, $c_0 > 0$ ($i = 0, 1, \dots, N-1$), and

$$a_i \leq b_i + c_0 \sum_{k=i+1}^{N-1} a_k.$$

Then

$$(4.6) \quad a_i \leq b_i + c_0 \sum_{k=i+1}^{N-1} (1 + c_0)^{k-i-1} b_k.$$

(2) Suppose that b , K are positive constants, $\gamma = 1$ or 2 , and for any $k = 0, 1, \dots, N-1$, $k+1 \leq j \leq N-1$

$$(4.7) \quad \begin{aligned} a_{k,j} &\leq b a_{k,j+1} + b |\pi| a_{j,j} + b K |\pi|^\gamma, \\ a_{k,k} &\leq b a_{k,k+1} + b K |\pi|^\gamma. \end{aligned}$$

Then, the following holds true:

$$(4.8) \quad \begin{aligned} a_{k,k} &\leq b^{N-k-1}a_{k,N-1} + b^{N-k}|\pi| \sum_{l=0}^{N-k-3} (1+b|\pi|)^l a_{k+1+l,N-1} + bK|\pi|^\gamma \sum_{l=0}^{N-k-2} b^l(1+b|\pi|)^l, \\ a_{k,j} &\leq b^{N-1-j}a_{k,N-1} + b^{N-j}|\pi| \sum_{l=0}^{N-j-2} (1+b|\pi|)^l a_{j+l,N-1} + K|\pi|^\gamma \sum_{l=1}^{N-j-1} b^l(1+b|\pi|)^l. \end{aligned}$$

Proof. We prove (4.6) by induction,

$$\begin{aligned} a_{N-1} &\leq b_{N-1}; \\ a_{N-2} &\leq b_{N-2} + c_0 b_{N-1}; \\ a_{N-3} &\leq b_{N-3} + c_0 b_{N-2} + c_0(c_0 + 1)b_{N-1}; \\ a_{N-4} &\leq b_{N-4} + c_0 b_{N-3} + c_0(c_0 + 1)b_{N-2} + c_0(c_0 + 1)^2 b_{N-1}; \\ &\dots \\ a_i &\leq b_i + c_0 \sum_{k=i+1}^{N-1} (c_0 + 1)^{k-i-1} b_k. \end{aligned}$$

Hence we obtain (4.6). (4.8) can also be proved by induction. ■

The following Lemma is about the regularity of $Y^{\pi(\cdot)}(\cdot)$.

Lemma 4.5. *Suppose that (A1)–(A4) hold true. Then, for any $k = 0, 1, \dots, N-1$, $k \leq j \leq N-1$ and $t \in [t_j, t_{j+1}]$, there exists a constant C , depending only on L and T , such that*

$$(4.9) \quad \mathbb{E}(|Y^k(t) - Y^k(t_j)|^2 + |Y^k(t) - Y^k(t_{j+1})|^2) \leq C|\pi|.$$

Proof. For any $t \in [t_j, t_{j+1}]$, by ESDE (4.2), it is easy to see that

$$\begin{aligned} (4.10) \quad &\mathbb{E}|Y^k(t) - Y^k(t_j)|^2 \leq 2\mathbb{E} \int_{t_j}^t |f(t_k, s, x(s), Y^j(s), Z^k(s))|^2 ds (t - t_j) + 2\mathbb{E} \int_{t_j}^t |Z^k(s)|^2 ds \\ &\leq 8L^2 \mathbb{E} \int_{t_j}^t (|f(t_k, s, 0, 0, 0)|^2 + |x(s)|^2 + |Y^j(s)|^2 + |Z^k(s)|^2) ds (t - t_j) + 2\mathbb{E} \int_{t_j}^t |Z^k(s)|^2 ds \\ &\leq K(t - t_j) + 8L^2 \mathbb{E} \int_{t_j}^t |Y^j(s)|^2 ds (t - t_j) + (8L^2(t - t_j) + 2) \mathbb{E} \int_{t_j}^t |Z^k(s)|^2 ds. \end{aligned}$$

We now estimate $\mathbb{E}|Y^k(t)|^2$, for $k = 0, 1, \dots, N-1$ and $t \in [t_k, t_{k+1}]$, which appears on the right side of (4.10). By Itô's formula,

$$\begin{aligned} &\mathbb{E}|Y^k(t)|^2 + \mathbb{E} \int_t^{t_{k+1}} |Z^k(s)|^2 ds \\ &\leq \mathbb{E}|Y^k(t_{k+1})|^2 + \mathbb{E} \int_t^{t_{k+1}} \left((2L + 3L^2 + 1)|Y^k(s)|^2 ds + |f(t_k, s, 0, 0, 0)|^2 + |x(s)|^2 + \frac{1}{2}|Z^k(s)|^2 \right) ds \\ &\leq \mathbb{E}|Y^k(t_{k+1})|^2 + (2L + 3L^2 + 1) \mathbb{E} \int_t^{t_{k+1}} |Y^k(s)|^2 ds + (L^2 + K)(t_k - t) + \frac{1}{2} \mathbb{E} \int_t^{t_k} |Z^k(s)|^2 ds. \end{aligned}$$

Consequently, by Gronwall's inequality,

$$(4.11) \quad \mathbb{E}|Y^k(t)|^2 \leq e^{(2L+3L^2+1)(t_{k+1}-t)} \left(\mathbb{E}|Y^k(t_{k+1})|^2 + (L^2 + K)|\pi| \right).$$

Now, we estimate $\mathbb{E}|Y^k(t)|^2$, for any $t \in [t_j, t_{j+1}]$ ($k+1 \leq j \leq N-1$), which appears in (4.11). With the similar calculus to that of (4.11), one can obtain

$$\begin{aligned}
& \mathbb{E}|Y^k(t)|^2 + \mathbb{E} \int_t^{t_{j+1}} |Z^k(s)|^2 ds \\
& \leq \mathbb{E}|Y^k(t_{j+1})|^2 + 2\mathbb{E} \int_t^{t_{j+1}} |Y^k(s)|(|f(t_k, s, 0, 0, 0)| + L|x(s)| + L|Y^j(s)| + L|Z^k(s)|) ds \\
& \leq \mathbb{E}|Y^k(t_{j+1})|^2 + (4L^2 + 1)\mathbb{E} \int_t^{t_{j+1}} |Y^k(s)|^2 ds \\
& \quad + \mathbb{E} \int_t^{t_{j+1}} (|f(t_k, s, 0, 0, 0)|^2 + |x(s)|^2 + |Y^j(s)|^2 + \frac{1}{2}|Z^k(s)|^2) ds \\
& \leq \mathbb{E}|Y^k(t_{j+1})|^2 + (4L^2 + 1)\mathbb{E} \int_t^{t_{j+1}} |Y^k(s)|^2 ds \\
& \quad + (L^2 + K)(t_{j+1} - t) + \mathbb{E} \int_t^{t_{j+1}} |Y^j(s)|^2 ds + \frac{1}{2}\mathbb{E} \int_t^{t_{j+1}} |Z^k(s)|^2 ds.
\end{aligned}$$

Also, by Gronwall's inequality, one has,

$$(4.12) \quad \mathbb{E}|Y^k(t)|^2 \leq e^{(4L^2+1)(t_{j+1}-t)} \left(\mathbb{E}|Y^k(t_{j+1})|^2 + (L^2 + K)|\pi| + \mathbb{E} \int_t^{t_{j+1}} |Y^j(s)|^2 ds \right).$$

Set $\bar{L} = \max\{2L + 3L^2 + 1, 4L^2 + 1\}$, $\alpha = e^{\bar{L}|\pi|}$, $\bar{K} = L^2 + K$, and $J_{k,j} = \sup_{t_j \leq t < t_{j+1}} \mathbb{E}|Y^k(t)|^2$, $j \geq k+1$. Then, by (4.11), (4.12) and Lemma 4.4, it comes out

$$(4.13) \quad J_{k,j} \leq \alpha^{N-1-j} J_{k,N-1} + \alpha^{N-j} |\pi| \sum_{l=0}^{N-2-j} (1 + \alpha|\pi|)^l J_{j+l,N-1} + \bar{K}|\pi| \sum_{l=1}^{N-1-j} \alpha^l (1 + \alpha|\pi|)^l.$$

Since for all $k \leq N-2$,

$$\begin{aligned}
(4.14) \quad & J_{k,N-1} = \sup_{t_{N-1} \leq t \leq T} \mathbb{E}|Y^k(t)|^2 \\
& \leq e^{\bar{L}|\pi|} \left(\mathbb{E}|g(t_k, x(T))|^2 + \bar{K}|\pi| + \int_{t_{N-1}}^T |Y^{N-1}(s)|^2 ds \right) \\
& \leq e^{\bar{L}|\pi|} \left(L^2 T + L^2 \mathbb{E}|x(T)|^2 + \bar{K}|\pi| + |\pi| J_{N-1,N-1} \right),
\end{aligned}$$

and

$$\begin{aligned}
(4.15) \quad & J_{N-1,N-1} = \sup_{t_{N-1} \leq t \leq T} \mathbb{E}|Y^{N-1}(t)|^2 \leq e^{\bar{L}|\pi|} \left(\mathbb{E}|g(t_{N-1}, x(T))|^2 + \bar{K}|\pi| \right) \\
& \leq e^{\bar{L}|\pi|} \left(L^2 T + L^2 \mathbb{E}|x(T)|^2 + \bar{K}|\pi| \right) \leq C < \infty,
\end{aligned}$$

(4.14), together with (4.15), yields that, for all $k = 0, 1, \dots, N-1$,

$$(4.16) \quad J_{k,N-1} \leq C < \infty.$$

Now, we estimate the right side of (4.13) term by term.

$$(4.17) \quad \alpha^{N-i-j} \leq \alpha^N = e^{\bar{L}|\pi|N} = e^{\bar{L}T} < \infty,$$

$$\begin{aligned}
(4.18) \quad & \alpha^{N-j}|\pi| \sum_{l=0}^{N-2-j} (1+\alpha|\pi|)^l J_{j+l,N-1} \leq C e^{\bar{L}T} \frac{(1+\alpha|\pi|)^N}{\alpha} \\
& = C \left(1 + \frac{T e^{\bar{L}T}}{N}\right)^N \leq C e^{T e^{\bar{L}T}} < \infty,
\end{aligned}$$

and

$$\begin{aligned}
(4.19) \quad & \bar{K}|\pi| \sum_{l=1}^{N-1-j} \alpha^l (1+\alpha|\pi|)^l \leq \bar{K}|\pi| \frac{\alpha^N (1+\alpha|\pi|)^N}{\alpha + \alpha^2|\pi| - 1} \\
& \leq \bar{K}|\pi| \frac{\alpha^N (1+\alpha|\pi|)^N}{\bar{L}|\pi|} \leq \frac{\bar{K}}{\bar{L}} e^{\bar{L}T} e^{T e^{\bar{L}T}} < \infty.
\end{aligned}$$

Hence, (4.13), together with (4.17)–(4.19), leads to, for any $k = 0, 1, \dots, N-1$ and $j \geq k+1$,

$$(4.20) \quad J_{k,j} = \sup_{t_j \leq t < t_{j+1}} \mathbb{E}|Y^k(t)|^2 < \infty.$$

Thereafter

$$J_{k,k} \leq \alpha J_{k,k+1} + \alpha \bar{K}|\pi| < \infty.$$

Furthermore, the second term of the right side in (4.10) turns into

$$(4.21) \quad \mathbb{E} \int_{t_j}^t |Y^j(s)|^2 ds (t - t_j) \leq C |t - t_j|^2 \leq C |t - t_j|.$$

Now, we need to estimate $\mathbb{E}|Z^k(t)|^2$, for any $t \in [t_j, t_{j+1}]$, which appears on the third term of the right side in (4.10). Since $(D_\theta Y^{\pi(\cdot)}(\cdot), D_\theta Z^{\pi(\cdot)}(\cdot))$, the Malliavin derivative of $(Y^{\pi(\cdot)}(\cdot), Z^{\pi(\cdot)}(\cdot))$, satisfies the following BSDE ([8, Proposition 5.3]):

$$\begin{aligned}
(4.22) \quad & \begin{cases} D_\theta Y^k(t_{j+1}) - D_\theta Y^k(t) \\ = \int_t^{t_{j+1}} \left(-f_x(t_k, s, x(s), Y^j(s), Z^k(s)) D_\theta x(s) - f_y(t_k, s, x(s), Y^j(s), Z^k(s)) D_\theta Y^j(s) \right. \\ \quad \left. - f_z(t_k, s, x(s), Y^j(s), Z^k(s)) D_\theta Z^k(s) \right) ds + \int_t^{t_{j+1}} D_\theta Z^k(s) dW(s), \quad t \in [t_j, t_{j+1}], \theta \in [0, t], \\ Z^k(t) = D_t Y^k(t), t \in [t_k, T]. \end{cases}
\end{aligned}$$

By Itô's formula, for any $t \in [t_k, t_{k+1}]$, one has

$$\begin{aligned}
& \mathbb{E}|D_\theta Y^k(t)|^2 + \mathbb{E} \int_t^{t_{k+1}} |D_\theta Z^k(s)|^2 ds \\
& \leq \mathbb{E} \left\{ |D_\theta Y^k(t_{k+1})|^2 + (2L + 3L^2) \int_t^{t_{k+1}} |D_\theta Y^k(s)|^2 ds \right. \\
& \quad \left. + \int_t^{t_{k+1}} |D_\theta x(s)|^2 ds + \frac{1}{2} \int_t^{t_{k+1}} |D_\theta Z^k(s)|^2 ds \right\},
\end{aligned}$$

by Gronwall's inequality, which deduces that,

$$\mathbb{E}|D_\theta Y^k(t)|^2 \leq e^{(2L+3L^2)(t_{k+1}-t)} \left(\mathbb{E}|D_\theta Y^k(t_{k+1})|^2 + K|\pi| \right).$$

Similarly, for any $t \in [t_j, t_{j+1}]$,

$$\begin{aligned}
& \mathbb{E}|D_\theta Y^k(t)|^2 + \mathbb{E} \int_t^{t_{j+1}} |D_\theta Z^k(s)|^2 ds \\
(4.23) \quad & \leq \mathbb{E} \left\{ |D_\theta Y^k(t_{j+1})|^2 + 4L^2 \int_t^{t_{j+1}} |D_\theta Y^k(s)|^2 ds \right. \\
& \quad \left. + \int_t^{t_{j+1}} (|D_\theta x(s)|^2 + |D_\theta Y^j(s)|^2) ds + \frac{1}{2} \int_t^{t_{j+1}} |D_\theta Z^k(s)|^2 ds \right\}.
\end{aligned}$$

Therefore, similar to (4.20), by virtue of Lemma 4.4, we has

$$(4.24) \quad \sup_{t_j \leq t \leq t_{j+1}} \mathbb{E}|D_\theta Y^k(t)|^2 < \infty.$$

By setting $\theta = t$ in (4.24), one gets, for any $t \in [t_k, T]$,

$$(4.25) \quad \mathbb{E}|Z^k(t)|^2 = \mathbb{E}|D_t Y^k(t)|^2 < \infty.$$

(4.10), together with (4.21) and (4.25), yields that

$$(4.26) \quad \mathbb{E}|Y^k(t) - Y^k(t_j)|^2 \leq C|t - t_j| \leq C|\pi|.$$

Similarly, we can get

$$\mathbb{E}|Y^k(t) - Y^k(t_{j+1})|^2 \leq C|t - t_{j+1}| \leq C|\pi|.$$

That completes the proof. ■

With this result at hand we can conclude:

Proposition 4.6. *Suppose that (A1)–(A4) hold true. Then, there exists a constant K , such that, for any $k = 0, 1, \dots, N-1$,*

$$\mathbb{E}|Y(t_k) - Y^k(t_k)|^2 + \mathbb{E} \int_{t_k}^T |Z(t_k, s) - Z^k(s)|^2 ds \leq K|\pi|.$$

Proof. Setting $h(t, s, z) = f(t, s, x(s), Y(s), z)$ and $\bar{h}(t, s, z) = f(\tau(t), s, x(s), Y^{\pi(s)}(s), z)$, by [17, Corrolary 3.6], we have

$$\begin{aligned}
& \mathbb{E}|Y(t_k) - Y^k(t_k)|^2 + \mathbb{E} \int_{t_k}^T |Z(t_k, s) - Z^k(s)|^2 ds \\
& \leq K \mathbb{E} \left(\int_{t_k}^T |f(t_k, s, x(s), Y(s), Z(t_k, s)) - f(\tau(t_k), s, x(s), Y^{\pi(s)}(s), Z(t_k, s))| ds \right)^2 \\
(4.27) \quad & \leq K \sum_{j=k}^{N-1} \mathbb{E} \int_{t_j}^{t_{j+1}} (|Y(s) - Y(t_j)|^2 + |Y^j(s) - Y^j(t_j)|^2 + |Y(t_j) - Y^j(t_j)|^2) ds \\
& \leq K|\pi| + K|\pi| \sum_{j=k+1}^{N-1} \mathbb{E}|Y(t_j) - Y^j(t_j)|^2 + K|\pi| \mathbb{E}|Y(t_k) - Y^k(t_k)|^2.
\end{aligned}$$

Here we apply Lemma 2.4 and Lemma 4.5. Taking $N > 2KT$ ($N \in \mathbb{N}$), then $K|\pi| \leq \frac{1}{2}$, and

$$(4.28) \quad \frac{1}{2} \mathbb{E}|Y(t_k) - Y^k(t_k)|^2 \leq K|\pi| + K|\pi| \sum_{j=k+1}^{N-1} \mathbb{E}|Y(t_j) - Y^j(t_j)|^2.$$

Denote $a_k = \mathbb{E}|Y(t_k) - Y^k(t_k)|^2$, $b_i = 2K|\pi|$ and $c_0 = 2K|\pi|$. Therefore, by Lemma 4.4, for any $k = 0, 1, \dots, N-1$,

$$(4.29) \quad \begin{aligned} \mathbb{E}|Y(t_k) - Y^k(t_k)|^2 &= a_k \leq b_k + c_0 \sum_{i=k+1}^{N-1} (1 + c_0)^{i-k-1} b_i \\ &\leq 2K|\pi| + 2K|\pi|(c_0 + 1)^N \leq 2K|\pi| + 2K|\pi|e^{2KT} \leq K|\pi|. \end{aligned}$$

Combining (4.27) with (4.29), we can have $\mathbb{E} \int_{t_k}^T |Z(t_k, s) - Z^k(s)|^2 ds \leq K|\pi|$. That completes the proof. \blacksquare

In the following part, we mainly provide the regularity of $Z^{\pi(\cdot)}(\cdot)$. Such a regularity, combining with that for $x(\cdot)$ and $Y^{\pi(\cdot)}(\cdot)$, can derive the rate of convergence of the Euler method (4.1). We present that regularity in two different cases: (I) $f = f(t, s, x, y)$; (II) $f = f(t, s, x, z)$. Here, we borrow some idea from [9].

Lemma 4.7. *Suppose that $f = f(t, s, x, y)$ in BSDE (4.2), and (A1)–(A4) hold true. Then, for any $k = 0, 1, \dots, N-1$, $k \leq j \leq N-1$ and $s \in [t_j, t_{j+1}]$, there exists a constant C , such that*

$$(4.30) \quad \mathbb{E}|Z^k(s) - Z^k(t_j)|^2 \leq K|\pi|.$$

Proof. We divide the proof into two steps.

Step 1. For any $k = 0, 1, \dots, N-1$, $k \leq j \leq N-1$ and $s \in [t_j, t_{j+1}]$, by (4.22), one gets

$$(4.31) \quad \begin{aligned} Z^k(s) - Z^k(t_j) &= D_s Y^k(s) - D_{t_j} Y^k(t_j) \\ &= (D_s Y^k(s) - D_{t_j} Y^k(s)) + (D_{t_j} Y^k(s) - D_{t_j} Y^k(t_j)). \end{aligned}$$

We claim that

$$(4.32) \quad \mathbb{E}|D_s Y^k(s) - D_{t_j} Y^k(s)|^2 \leq K|\pi|.$$

Indeed, by (4.22), for $\theta_1, \theta_2 \in [t_j, t_{j+1}]$, $\theta_2 \leq \theta_1 \leq s$,

$$\begin{aligned} &\mathbb{E}|D_{\theta_1} Y^k(s) - D_{\theta_2} Y^k(s)|^2 + \mathbb{E} \int_s^{t_{j+1}} |D_{\theta_1} Z^k(t) - D_{\theta_2} Z^k(t)|^2 dt \\ &= \mathbb{E}|D_{\theta_1} Y^k(t_{j+1}) - D_{\theta_2} Y^k(t_{j+1})|^2 \\ &\quad + 2\mathbb{E} \int_s^{t_{j+1}} \left\langle D_{\theta_1} Y^k(t) - D_{\theta_1} Y^k(t), f_x(D_{\theta_1} x(t) - D_{\theta_2} x(t)) + f_y(D_{\theta_1} Y^j(t) - D_{\theta_2} Y^j(t)) \right\rangle dt \\ &\leq \mathbb{E}|D_{\theta_1} Y^k(t_{j+1}) - D_{\theta_2} Y^k(t_{j+1})|^2 + 2L^2 \mathbb{E} \int_s^{t_{j+1}} |D_{\theta_1} Y^k(t) - D_{\theta_2} Y^k(t)|^2 dt \\ &\quad + \mathbb{E} \int_s^{t_{j+1}} |D_{\theta_1} x(t) - D_{\theta_2} x(t)|^2 dt + \mathbb{E} \int_s^{t_{j+1}} |D_{\theta_1} Y^j(t) - D_{\theta_2} Y^j(t)|^2 dt. \end{aligned}$$

Hence, by Lemma 2.2 and Gronwall's inequality, one has, for any $s \in [t_j, t_{j+1}]$,

$$\mathbb{E}|D_{\theta_1} Y^k(s) - D_{\theta_2} Y^k(s)|^2$$

$$\begin{aligned}
&\leq e^{2L^2(t_{j+1}-s)} \left(\mathbb{E} |D_{\theta_1} Y^k(t_{j+1}) - D_{\theta_2} Y^k(t_{j+1})|^2 + K|\theta_1 - \theta_2|(t_{j+1} - s) \right. \\
&\quad \left. + \mathbb{E} \int_s^{t_{j+1}} |D_{\theta_1} Y^j(t) - D_{\theta_2} Y^j(t)|^2 dt \right) \\
&\leq e^{2L^2(t_{j+1}-s)} \left(\mathbb{E} |D_{\theta_1} Y^k(t_{j+1}) - D_{\theta_2} Y^k(t_{j+1})|^2 + K|\pi|^2 \right. \\
&\quad \left. + \mathbb{E} \int_s^{t_{j+1}} |D_{\theta_1} Y^j(t) - D_{\theta_2} Y^j(t)|^2 dt \right).
\end{aligned}$$

Similarly, for any $s \in [t_k, t_{k+1}]$,

$$\mathbb{E} |D_{\theta_1} Y^k(s) - D_{\theta_2} Y^k(s)|^2 \leq e^{2L^2(t_{k+1}-s)} \left(\mathbb{E} |D_{\theta_1} Y^k(t_{k+1}) - D_{\theta_2} Y^k(t_{k+1})|^2 + K|\pi|^2 \right).$$

By Lemma 4.4, with the similar procedure used in the proof of Lemma 4.5, one can get, for any $s \in [t_j, t_{j+1}]$,

$$(4.33) \quad \sup_{t_j \leq s \leq t_{j+1}} \mathbb{E} |D_{\theta_1} Y^k(s) - D_{\theta_2} Y^k(s)|^2 \leq K|\pi|.$$

Setting $\theta_1 = s$, $\theta_2 = t_j$, one easily obtains (4.32).

Step 2. We claim that, for any $s \in [t_j, t_{j+1}]$,

$$(4.34) \quad \mathbb{E} |D_{t_j} Y^k(s) - D_{t_j} Y^k(t_j)| \leq K|s - t_j|.$$

For any $\theta \leq t_j$, $t \in [t_j, t_{j+1}]$, by virtue of Eq. (4.2),

$$D_{\theta} Y^k(t) = \mathbb{E} \left(D_{\theta} Y^k(T) + \int_t^T F(s) ds \middle| \mathcal{F}_t \right),$$

where

$$\begin{aligned}
\int_t^T F(s) ds &= \int_t^{t_{j+1}} f_x(t_k, s, x(s), Y^j(s)) D_{\theta} x(s) + f_y(t_k, s, x(s), Y^j(s)) D_{\theta} Y^j(s) ds \\
&\quad + \sum_{l=j+1}^{N-1} \int_{t_l}^{t_{l+1}} f_x(t_k, s, x(s), Y^l(s)) D_{\theta} x(s) + f_y(t_k, s, x(s), Y^l(s)) D_{\theta} Y^l(s) ds.
\end{aligned}$$

Then

$$\begin{aligned}
&D_{\theta} Y^k(t) - D_{\theta} Y^k(t_j) \\
(4.35) \quad &= \left\{ \mathbb{E} (D_{\theta} Y^k(T) | \mathcal{F}_t) - \mathbb{E} (D_{\theta} Y^k(T) | \mathcal{F}_{t_j}) \right\} + \left\{ \mathbb{E} \left(\int_t^T F(s) ds | \mathcal{F}_t \right) - \mathbb{E} \left(\int_t^T F(s) ds | \mathcal{F}_{t_j} \right) \right\} \\
&:= I_1 + I_2.
\end{aligned}$$

For I_1 , since

$$D_{\theta} Y^k(T) = \mathbb{E} D_{\theta} Y^k(T) + \int_0^T \mathbb{E} (D_s D_{\theta} Y^k(T) | \mathcal{F}_s) dW(s),$$

by Lemma 2.2, one can have

$$\begin{aligned}
&\mathbb{E} I_1^2 = \mathbb{E} \int_{t_j}^t \left| \mathbb{E} (D_s D_{\theta} Y^k(T) | \mathcal{F}_s) \right|^2 ds \leq \mathbb{E} \int_{t_j}^t |D_s D_{\theta} Y^k(T)|^2 ds \\
(4.36) \quad &= \mathbb{E} \int_{t_j}^t |D_s D_{\theta} g(t_k, x(T))|^2 ds = \mathbb{E} \int_{t_j}^t |g_{xx} D_{\theta} x(T) D_u x(T) + g_x D_s D_{\theta} x(T)|^2 ds \leq K|t - t_j|.
\end{aligned}$$

For I_2 ,

$$\begin{aligned}
(4.37) \quad I_2 &= \left\{ \mathbb{E} \left(\int_t^T F(s) ds \middle| \mathcal{F}_t \right) - \mathbb{E} \left(\int_{t_j}^T F(s) ds \middle| \mathcal{F}_t \right) \right\} \\
&\quad + \left\{ \mathbb{E} \left(\int_{t_j}^T F(s) ds \middle| \mathcal{F}_t \right) - \mathbb{E} \left(\int_{t_j}^T F(s) ds \middle| \mathcal{F}_{t_j} \right) \right\} \\
&:= I_{21} + I_{22}.
\end{aligned}$$

By Lemma 2.2 and (4.24), it is easy to check that

$$\begin{aligned}
(4.38) \quad \mathbb{E} I_{21}^2 &= \mathbb{E} \left| \int_t^{t_{j+1}} (f_x(t_k, s, x(s), Y^j(s)) D_\theta x(s) + f_y(t_k, s, x(s), Y^j(s)) D_\theta Y^j(s)) ds \right|^2 \\
&\leq K(t - t_j) \mathbb{E} \int_{t_j}^t (|D_\theta x(s)|^2 + |D_\theta Y^j(s)|^2) ds \leq K|t - t_j|.
\end{aligned}$$

For the I_{22} part, by Clark-Ocone representation formula,

$$\int_{t_j}^T F(s) ds = \mathbb{E} \left(\int_{t_j}^T F(s) ds \right) + \int_0^T \mathbb{E} \left(D_u \int_{t_j}^T F(s) ds \middle| \mathcal{F}_u \right) dW(u),$$

it admits the following representation:

$$I_{22} = \int_{t_j}^t \mathbb{E} \left(D_u \int_{t_j}^T F(s) ds \middle| \mathcal{F}_u \right) dW(u).$$

It is easy to check that

$$\begin{aligned}
&D_u \int_{t_j}^T F(s) ds = \int_{t_j}^T D_u F(s) ds \\
&= \sum_{l=j}^{N-1} \int_{t_l}^{t_{l+1}} D_u (f_x(t_k, s, x(s), Y^l(s)) D_\theta x(s) + f_y(t_k, s, x(s), Y^l(s)) D_\theta Y^l(s)) ds \\
&= \sum_{l=j}^{N-1} \int_{t_l}^{t_{l+1}} (f_{xx} D_\theta x(s) D_u x(s) + f_{xy} D_\theta x(s) D_u Y^l(s) + f_x D_u D_\theta x(s) \\
&\quad + f_{yx} D_\theta Y^l(s) D_u x(s) + f_{yy} D_\theta Y^l(s) D_u Y^l(s) + f_y D_u D_\theta Y^l(s)) ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(4.39) \quad &\mathbb{E} \left| D_u \int_{t_j}^T F(s) ds \right|^2 \leq T \mathbb{E} \int_{t_j}^T |D_u F(s)|^2 ds \\
&\leq K \sum_{l=j}^{N-1} \mathbb{E} \int_{t_l}^{t_{l+1}} (|D_\theta x(s) D_u x(s)|^2 + |D_\theta x(s) D_u Y^l(s)|^2 + |D_u D_\theta x(s)|^2 \\
&\quad + |D_\theta Y^l(s) D_u x(s)|^2 + |D_\theta Y^l(s) D_u Y^l(s)|^2 + |D_u D_\theta Y^l(s)|^2) ds \\
&\leq K \sum_{l=j}^{N-1} \mathbb{E} \int_{t_l}^{t_{l+1}} (|D_\theta x(s)|^4 + |D_u x(s)|^4 + |D_u Y^l(s)|^4 + |D_\theta Y^l(s)|^4 \\
&\quad + |D_u D_\theta x(s)|^2 + |D_u D_\theta Y^l(s)|^2) ds.
\end{aligned}$$

Now, we estimate each term on the right side of the above inequality. By Itô's formula,

$$\begin{aligned}
& \mathbb{E}|D_\theta Y^k(t)|^4 + 6\mathbb{E} \int_t^{t_{j+1}} |D_\theta Y^k(s)|^2 |D_\theta Z^k(s)|^2 ds \\
&= \mathbb{E}|D_\theta Y^k(t_{j+1})|^4 + 4\mathbb{E} \int_t^{t_{j+1}} |D_\theta Y^k(s)|^2 \langle D_\theta Y^k(s), f_x D_\theta x(s) + f_y D_\theta Y^j(s) \rangle ds \\
&\leq \mathbb{E}|D_\theta Y^k(t_{j+1})|^4 + 4L\mathbb{E} \int_t^{t_{j+1}} \left(\frac{3}{2} |D_\theta Y^k(s)|^4 + 4|D_\theta x(s)|^4 + 4|D_\theta Y^j(s)|^4 \right) ds \\
&\leq \mathbb{E}|D_\theta Y^k(t_{j+1})|^4 + 6L\mathbb{E} \int_t^{t_{j+1}} |D_\theta Y^k(s)|^4 ds + K|\pi| + K\mathbb{E} \int_t^{t_{j+1}} |D_\theta Y^j(s)|^4 ds.
\end{aligned}$$

Thus, by Lemma 4.4, one get

$$(4.40) \quad \sup_{t_j \leq t \leq t_{j+1}} \mathbb{E}|D_\theta Y^k(t)|^4 < \infty.$$

For any $u \leq t_j \leq t \leq t_{j+1}$,

$$\begin{aligned}
& D_u D_\theta Y^k(t_{j+1}) - D_u D_\theta Y^k(t) \\
&= \int_t^{t_{j+1}} \left(f_{xx} D_\theta x(s) D_u x(s) + f_{xy} D_\theta x(s) D_u Y^j(s) + f_x D_u D_\theta x(s) \right. \\
&\quad \left. + f_{yx} D_\theta Y^j(s) D_u x(s) + f_{yy} D_\theta Y^j(s) D_u Y^j(s) + f_y D_u D_\theta Y^j(s) \right) ds \\
&\quad + \int_t^{t_{j+1}} D_u D_\theta Z^k(s) dW(s).
\end{aligned} \tag{4.41}$$

Hence, by Itô's formula,

$$\begin{aligned}
& \mathbb{E}|D_u D_\theta Y^k(t)|^2 + \mathbb{E} \int_t^{t_{j+1}} |D_u D_\theta Z^k(s)|^2 ds \\
&\leq \mathbb{E}|D_u D_\theta Y^k(t_{j+1})|^2 + \mathbb{E} \int_t^{t_{j+1}} \left(6L^2 |D_u D_\theta Y^k(s)|^2 + |D_\theta x(s) D_u x(s)|^2 + |D_\theta x(s) D_u Y^j(s)|^2 \right. \\
&\quad \left. + |D_u D_\theta x(s)|^2 + |D_\theta Y^j(s) D_u x(s)|^2 + |D_\theta Y^j(s) D_u Y^j(s)|^2 + |D_u D_\theta Y^j(s)|^2 \right) ds \\
&\leq \mathbb{E}|D_u D_\theta Y^k(t_{j+1})|^2 + 6L^2 \mathbb{E} \int_t^{t_{j+1}} |D_u D_\theta Y^k(s)|^2 ds + K|\pi| + \mathbb{E} \int_t^{t_{j+1}} |D_u D_\theta Y^j(s)|^2 ds
\end{aligned}$$

Also, by Lemma 4.4, we have

$$(4.42) \quad \sup_{t_j \leq t \leq t_{j+1}} \mathbb{E}|D_u D_\theta Y^k(t)|^2 < \infty.$$

Therefore, (4.39), together with (4.40) and (4.42), yields that

$$\mathbb{E} \left| D_u \int_{t_j}^T F(s) ds \right|^2 < \infty.$$

Furthermore,

$$(4.43) \quad \mathbb{E}|I_{22}|^2 = \mathbb{E} \int_{t_j}^t \left| \mathbb{E} \left(D_u \int_{t_j}^T F(s) ds \middle| \mathcal{F}_u \right) \right|^2 du \leq \mathbb{E} \int_{t_j}^t \left| D_u \int_{t_j}^T F(s) ds \right|^2 du \leq K|t - t_j|.$$

Finally, by (4.35)–(4.38) and (4.43), one gets

$$\mathbb{E}|D_\theta Y_k(t) - D_\theta Y_k(t_j)|^2 \leq K|t - t_j|,$$

which deduces (4.34) by setting $\theta = t_j$. Now combining (4.31) with (4.32) and (4.34), we have the regularity of Z (4.30). \blacksquare

The following regularity of $Z^{\pi(\cdot)}(\cdot)$ is in the case: $f = f(t, s, x, z)$.

Lemma 4.8. *Suppose that $f = f(t, s, x, z)$ in BSDE (4.2), and (A1)–(A4) hold true. Then, for any $k = 0, 1, \dots, N-1$, $k \leq j \leq N-1$ and $s \in [t_j, t_{j+1}]$, there exists a constant C , such that*

$$(4.44) \quad \mathbb{E}|Z^k(s) - Z^k(t_j)|^2 \leq C|\pi|.$$

We need the following lemma to prove the above result.

Lemma 4.9. *Let (A1) hold, and for any $k = 0, 1, \dots, N-1$, $\Psi_k(\cdot)$ and $\Phi_k(\cdot)$ solve the following SDEs*

$$(4.45) \quad \begin{cases} d\Psi_k(t) = \Psi_k(t)f_z(t_k, t, x(t), Z^k(t))dW(t), & t \in [0, T], \\ \Psi(0) = I_n \end{cases}$$

and

$$(4.46) \quad \begin{cases} d\Phi_k(t) = (f_z(t_k, t, x(t), Z^k(t)))^2 \Phi_k(t)dt \\ \quad - f_z(t_k, t, x(t), Z^k(t))\Phi_k(t)dW(t), & t \in [0, T], \\ \Phi(0) = I_n, \end{cases}$$

respectively. Then, for any $p \geq 2$,

$$(4.47) \quad \mathbb{E}\left(\sup_{0 \leq t \leq T} |\Psi_k(t)|^p\right) + \mathbb{E}\left(\sup_{0 \leq t \leq T} |\Phi_k(t)|^p\right) \leq C,$$

$$(4.48) \quad \mathbb{E}\left(\sup_{s \leq t \leq T} |\Phi_k(s)\Psi_k(t)|^p\right) + \mathbb{E}\left(\sup_{s \leq t \leq T} |\Phi_k(t)\Psi_k(s)|^p\right) \leq C,$$

$$(4.49) \quad \mathbb{E}|(\Phi_k(t) - \Phi_k(s))\Psi_k(T_0)|^p \leq C|t - s|^{\frac{p}{2}}, \quad t, s \leq T_0 \leq T;$$

and for any $p \in [2, 2p_0]$,

$$(4.50) \quad \mathbb{E}\left(\sup_{\theta, s \leq t \leq T} |D_\theta(\Phi_k(s)\Psi_k(t))|^p\right) \leq C,$$

where C depends only on p , L and T .

Proof. First of all, for any $x_0 \in \mathbb{R}^n$, set $x(\cdot) = \Psi_k^\top(\cdot)x_0$. Then $x(\cdot)$ solves the following SDE:

$$\begin{cases} dx(t) = f_z^\top(t)x(t)dW(t), & t \in [0, T], \\ x(0) = x_0. \end{cases}$$

Then, by Lemma 2.1,

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |x(t)|^p\right) \leq C|x_0|^p.$$

Consequently,

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Psi(t)|^p\right) = \sup_{x_0 \in \mathbb{R}^n} \frac{\mathbb{E}\left(\sup_{0 \leq t \leq T} |x(t)|^p\right)}{|x_0|^p} \leq C.$$

Here C depends only on p, L and T . Similarly, one can prove $\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Phi_k(t)|^p\right) \leq C$, and then (4.47) is proved.

Next, we only prove the second part $\mathbb{E}\left(\sup_{s \leq t \leq T} |\Phi_k(t)\Psi_k(s)|^p\right) \leq C$ of (4.48). The first one can be proved with the similar procedure. For any $x_0 \in \mathbb{R}^n$, set $x_s(t) = \Phi_k(t)\Psi_k(s)x_0$. Then $x_s(t)$ solves the following SDE:

$$\begin{cases} dx_s(t) = (f_z)^2 x_s(t) dt - f_z x_s(t) dW(t), & t \in [s, T], \\ x_s(s) = x_0. \end{cases}$$

Then, also by Lemma 2.1,

$$\mathbb{E}\left(\sup_{s \leq t \leq T} |\Phi_k(t)\Psi_k(s)x_0|^p\right) = \mathbb{E}\left(\sup_{s \leq t \leq T} |x_s(t)|^p\right) \leq C|x_0|^p,$$

where C depends only on p, L and T .

Now, by Eq. (4.46), one has

$$\begin{aligned} & \mathbb{E}|\Phi_k(t) - \Phi_k(s))\Psi_k(T_0)|^p \\ &= \mathbb{E}\left|\int_s^t (f_z(\tau))^2 \Phi_k(\tau) d\tau \Psi_k(T_0) + \int_s^t f_z(\tau) \Phi_k(\tau) dW(\tau) \Psi_k(T_0)\right|^p \\ (4.51) \quad & \leq C\mathbb{E}\left(\int_s^t |\Phi_k(\tau)\Psi_k(T_0)| d\tau\right)^p + C\mathbb{E}\left|\int_s^t f_z(\tau) \Phi_k(\tau) dW(\tau) \Psi_k(T_0)\right|^p \\ & := CJ_1 + CJ_2. \end{aligned}$$

For J_1 , by (4.48), we have

$$\begin{aligned} J_1 &\leq \mathbb{E} \int_s^t |\Phi_k(\tau)\Psi_k(T_0)|^p d\tau \left(\int_s^t 1 d\tau\right)^{p-1} \\ (4.52) \quad &= \int_s^t \mathbb{E} |\Phi_k(\tau)\Psi_k(T_0)|^p d\tau (t-s)^{p-1} \leq C(t-s)^p, \end{aligned}$$

where C depends only on p, L and T . For J_2 , by (4.48), Hölder's inequality and Burkholder-Davis-Gundy inequality, we also can obtain

$$\begin{aligned} J_2 &= \mathbb{E}\left|\int_s^t f_z(\tau) \Phi_k(\tau) dW(\tau) \Psi_k(T_0)\right|^p = \mathbb{E}\left|\int_s^t f_z(\tau) \Phi_k(\tau) \Psi_k(s) dW(\tau) \Phi_k(s) \Psi_k(T_0)\right|^p \\ &\leq \left(\mathbb{E}\left|\int_s^t f_z(\tau) \Phi_k(\tau) \Psi_k(s) dW(\tau)\right|^{2p}\right)^{1/2} \left(\mathbb{E}|\Phi_k(s)\Psi_k(T_0)|^{2p}\right)^{1/2} \\ (4.53) \quad &\leq C\left(\mathbb{E}\left(\int_s^t |\Phi_k(\tau)\Psi_k(s)|^2 d\tau\right)^p\right)^{1/2} \\ &\leq C\left\{\mathbb{E}\left[\left(\int_s^t 1 d\tau\right)^{\frac{p-1}{p}} \left(\int_s^t |\Phi_k(\tau)\Psi_k(s)|^{2p} d\tau\right)^{\frac{1}{p}}\right]^p\right\}^{1/2} \\ &\leq C(t-s)^{\frac{p}{2}}, \end{aligned}$$

where C depends only on p, L . Combining (4.51)–(4.53), we have (4.49).

Finally, we prove (4.50). Indeed, For any $0 \leq \theta, s \leq t \leq T$, $D_\theta(\Phi_k(s)\Psi_k(\cdot))$ satisfies the following SDE:

$$\begin{cases} dD_\theta(\Phi_k(s)\Psi_k(t)) = \left(D_\theta(\Phi_k(s)\Psi_k(t))f_z(t_k, t, x(t), Z^k(t)) \right. \\ \quad \left. + (\Phi_k(s)\Psi_k(t))(f_{zx}D_\theta x(t) + f_{zz}D_\theta Z^k(t)) \right) dW(t), \quad \theta \leq t \leq T, \\ D_\theta(\Phi_k(s)\Psi_k(\theta)) = 0, \\ D_\theta(\Phi_k(s)\Psi_k(t)) = 0, \quad 0 \leq t < \theta. \end{cases}$$

For any $x_0 \in \mathbb{R}^n$, set $x_{\theta,s}(\cdot) = D_\theta(\Psi_k^\top(\cdot)\Phi_k^\top(s))x_0$ and $y_s(\cdot) = \Psi_k^\top(\cdot)\Phi_k^\top(s)x_0$. Then $x_{\theta,s}(\cdot)$ satisfies the following SDE:

$$\begin{cases} dx_{\theta,s}(t) = (f_z^\top(t)x_{\theta,s}(t) + (D_\theta x^\top(t)f_{zx}^\top(t) + D_\theta Z^{k\top}(t)f_{zz}^\top(t))y_s(t))dW(t), \quad \theta \leq t \leq T, \\ x_{\theta,s}(\theta) = 0, \\ x_{\theta,s}(t) = 0, \quad 0 \leq t < \theta \leq T. \end{cases}$$

For any $p \in [2, 2p_0)$, by Lemma 2.1, we have

$$\begin{aligned} & \mathbb{E} \left(\sup_{\theta \leq t \leq T} |D_\theta(\Psi_k^\top(t)\Phi_k^\top(s))x_0|^p \right) \\ & \leq C \mathbb{E} \left(\int_\theta^T |(D_\theta x^\top(t)f_{zx}^\top(t) + D_\theta Z^{k\top}(t)f_{zz}^\top(t))y_s(t)|^2 dt \right)^{\frac{p}{2}} \\ (4.54) \quad & \leq C \mathbb{E} \left(\int_\theta^T |D_\theta x(t)|^2 |y_s(t)|^2 dt \right)^{\frac{p}{2}} \\ & \quad + C \left\{ \mathbb{E} \left(\int_\theta^T |D_\theta Z^k(t)|^2 dt \right)^{\frac{2p_0}{2}} \right\}^{\frac{p}{2p_0}} \left(\mathbb{E} \left(\sup_{s \leq t \leq T} |y_s(t)|^{\frac{2pp_0}{2p_0-p}} \right) \right)^{\frac{2p_0-p}{2p_0}}, \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E} \left(\int_\theta^T |D_\theta x(t)|^2 |y_s(t)|^2 dt \right)^{\frac{p}{2}} \leq \left\{ \mathbb{E} \left[\int_\theta^T |D_\theta x(t)|^{p_0} dt \left(\int_\theta^T |y_s(t)|^{\frac{2p_0}{p_0-2}} dt \right)^{\frac{p_0-2}{2}} \right] \right\}^{\frac{p}{p_0}} \\ (4.55) \quad & \leq C \left[\mathbb{E} \int_\theta^T |D_\theta x(t)|^{2p_0} dt \right]^{\frac{p}{2p_0}} \left[\mathbb{E} \left(\sup_{\theta \leq t \leq T} |y_s(t)|^{2p_0} \right) \right]^{\frac{p}{2p_0}} \\ & \leq C \left(\sup_{\theta \leq t \leq T} \mathbb{E} |D_\theta x(t)|^{2p_0} \right)^{\frac{p}{2p_0}} \left[\mathbb{E} \left(\sup_{\theta \leq t \leq T} |y_s(t)|^{2p_0} \right) \right]^{\frac{p}{2p_0}}, \end{aligned}$$

$$(4.56) \quad \mathbb{E} \left(\int_\theta^T |D_\theta Z^k(t)|^2 dt \right)^{\frac{2p_0}{2}} \leq C \left\{ \mathbb{E} |D_\theta x(T)|^{2p} + \mathbb{E} \left(\int_\theta^T |D_\theta x(t)| dt \right)^{2p_0} \right\},$$

where C depends only on p_0, L and T . (4.54), together with (4.55), (4.56) and (4.48), yields (4.50). That completes the proof. \blacksquare

Now, we can prove Lemma 4.8.

Proof of Lemma 4.8. We split the proof into two steps.

Step 1. Similar to (4.31), we also obtain: for any $k = 0, 1, \dots, N-1$, $k \leq j \leq N-1$ and $s \in [t_j, t_{j+1}]$,

$$(4.57) \quad \begin{aligned} Z^k(s) - Z^k(t_j) &= D_s Y^k(s) - D_{t_j} Y^k(t_j) \\ &= (D_s Y^k(s) - D_{t_j} Y^k(s)) + (D_{t_j} Y^k(s) - D_{t_j} Y^k(t_j)), \end{aligned}$$

and

$$(4.58) \quad \mathbb{E}|D_s Y^k(s) - D_{t_j} Y^k(s)|^2 \leq K|\pi|.$$

Step 2. We claim that, for any $s \in [t_j, t_{j+1}]$, there exists a constant K , such that

$$(4.59) \quad \mathbb{E}|D_{t_j} Y^k(s) - D_{t_j} Y^k(t_j)| \leq K|s - t_j|.$$

In order to do this, for any $\theta \leq t_j$, applying Itô's formula to $\Psi_k(\cdot)D_\theta Y^k(\cdot)$, we obtain

$$\begin{aligned} \Psi_k(t)D_\theta Y^k(t) &= \Psi_k(T)D_\theta Y^k(T) + \int_t^T \Psi_k(s)f_x D_\theta x(s)ds \\ &\quad + \int_t^T \Psi_k(s)(f_z D_\theta Y^k(s) + D_\theta Z^k(s))dW(s), \quad t \in [t_j, t_{j+1}], \quad j \geq k. \end{aligned}$$

Since $\Psi_k(\cdot)\Phi_k(\cdot) = I_n$, one can get

$$D_\theta Y^k(t) = \mathbb{E}\left(\Phi_k(t)\Psi_k(T)D_\theta Y^k(T) + \Phi_k(t) \int_t^T \Psi_k(s)f_x D_\theta x(s)ds \middle| \mathcal{F}_t\right).$$

Then

$$(4.60) \quad \begin{aligned} &D_\theta Y^k(t) - D_\theta Y^k(t_j) \\ &= \mathbb{E}(\Phi_k(t)\Psi_k(T)D_\theta Y^k(T) | \mathcal{F}_t) - \mathbb{E}(\Phi_k(t_j)\Psi_k(T)D_\theta Y^k(T) | \mathcal{F}_{t_j}) \\ &\quad + \mathbb{E}\left(\Phi_k(t) \int_t^T \Psi_k(s)f_x D_\theta x(s)ds \middle| \mathcal{F}_t\right) - \mathbb{E}\left(\Phi_k(t_j) \int_{t_j}^T \Psi_k(s)f_x D_\theta x(s)ds \middle| \mathcal{F}_{t_j}\right) \\ &:= I_1 + I_2. \end{aligned}$$

Now, we estimate I_1 and I_2 , respectively. I_1 can be written as

$$(4.61) \quad \begin{aligned} I_1 &= \mathbb{E}\left((\Phi_k(t) - \Phi_k(t_j))\Psi_k(T)D_\theta Y^k(T) \middle| \mathcal{F}_t\right) \\ &\quad + \mathbb{E}(\Phi_k(t_j)\Psi_k(T)D_\theta Y^k(T) | \mathcal{F}_t) - \mathbb{E}(\Phi_k(t_j)\Psi_k(T)D_\theta Y^k(T) | \mathcal{F}_{t_j}) \\ &:= I_{11} + I_{12}. \end{aligned}$$

By (4.49), a direct calculate leads to

$$(4.62) \quad \mathbb{E}|I_{11}|^2 \leq \left(\mathbb{E}|(\Phi_k(t) - \Phi_k(t_j))\Psi_k(T)|^4 \mathbb{E}|D_\theta Y^k(T)|^4\right)^{1/2} \leq C|t - t_j|.$$

Meanwhile, by Clark-Ocone representation formula,

$$\Phi_k(t_j)\Psi_k(T)D_\theta Y^k(T) = \mathbb{E}(\Phi_k(t_j)\Psi_k(T)D_\theta Y^k(T)) + \int_0^T u_\theta(s)dW(s),$$

where $u_\theta(\cdot) = \mathbb{E}\left(D_\theta(\Phi(t_j)\Psi_k(T))D_\theta Y^k(T) + \Phi(t_j)\Psi_k(T)D_\theta D_\theta Y^k(T)|\mathcal{F}_\cdot\right)$. Therefore, by Lemma 4.9, one gets

$$\begin{aligned}\mathbb{E}|u_\theta(s)|^2 &\leq C\left\{\left(\mathbb{E}|D_s(\Phi(t_j)\Psi_k(T))|^4\mathbb{E}|D_\theta Y^k(T)|^4\right)^{1/2} + \left(\mathbb{E}|\Phi_k(t_j)\Psi_k(T)|^4\mathbb{E}|D_s D_\theta Y^k(T)|^4\right)^{1/2}\right\} \\ &\leq C < \infty.\end{aligned}$$

Thus,

$$(4.63) \quad \mathbb{E}|I_{12}|^2 = \mathbb{E}\left|\int_{t_j}^t u_\theta(s)dW(s)\right|^2 = \mathbb{E}\int_{t_j}^t |u_\theta(s)|^2 ds \leq C|t - t_j|.$$

For I_2 , we can rewrite it as follows:

$$\begin{aligned}(4.64) \quad I_2 &= \mathbb{E}\left(\int_t^T (\Phi_k(t) - \Phi_k(t_j))\Psi_k(s)f_x D_\theta x(s)ds|\mathcal{F}_t\right) \\ &\quad + \mathbb{E}\left(\Phi_k(t_j)\left(\int_t^T \Psi_k(s)f_x D_\theta x(s)ds - \int_{t_j}^T \Psi_k(s)f_x D_\theta x(s)ds\right)|\mathcal{F}_t\right) \\ &\quad + \mathbb{E}\left(\int_{t_j}^T \Phi_k(t_j)\Psi_k(s)f_x D_\theta x(s)ds|\mathcal{F}_t\right) - \mathbb{E}\left(\int_{t_j}^T \Phi_k(t_j)\Psi_k(s)f_x D_\theta x(s)ds|\mathcal{F}_{t_j}\right) \\ &:= I_{21} + I_{22} + I_{23}.\end{aligned}$$

It is easy to check that

$$\begin{aligned}(4.65) \quad \mathbb{E}I_{21}^2 &\leq \left(\mathbb{E}\int_t^T |(\Phi_k(t) - \Phi_k(t_j))\Psi_k(s)|^4 ds\right)^{1/2} \left(\mathbb{E}\int_t^T |f_x D_\theta x(s)|^4 ds\right)^{1/2} \\ &\leq K(t - t_j) \left(\sup_{0 \leq s \leq T} \mathbb{E}|D_\theta x(s)|^4\right)^{1/2} \\ &\leq K|t - t_j|,\end{aligned}$$

and

$$\begin{aligned}(4.66) \quad \mathbb{E}I_{22}^2 &\leq |t - t_j| \mathbb{E}\int_{t_j}^t |\Phi_k(t_j)\Psi_k(s)|^2 |f_x D_\theta x(s)|^2 ds \\ &\leq K(t - t_j) \left(\mathbb{E}\int_{t_j}^t |\Phi_k(t_j)\Psi_k(s)|^4 ds\right)^{1/2} \left(\mathbb{E}\int_{t_j}^t |f_x D_\theta x(s)|^4 ds\right)^{1/2} \\ &\leq K|t - t_j|.\end{aligned}$$

Now, we are in the step to estimate I_{23} . By Clark-Ocone representation formula,

$$\int_{t_j}^T \Phi_k(t_j)\Psi_k(s)f_x D_\theta x(s)ds = \mathbb{E}\int_{t_j}^T \Phi_k(t_j)\Psi_k(s)f_x D_\theta x(s)ds + \int_0^T v_\theta(u)dW(u),$$

where

$$\begin{aligned}(4.67) \quad &v_\theta(u) \\ &= \mathbb{E}\left(D_u \int_{t_j}^T \Phi_k(t_j)\Psi_k(s)f_x D_\theta x(s)ds|\mathcal{F}_u\right) \\ &= \mathbb{E}\left(\int_{t_j}^T \Phi_k(t_j)D_u \Psi_k(s)f_x D_\theta x(s)ds|\mathcal{F}_u\right) + \mathbb{E}\left(\int_{t_j}^T \Phi_k(t_j)\Psi_k(s)D_u(f_x D_\theta x(s))ds|\mathcal{F}_u\right) \\ &:= V_1 + V_2.\end{aligned}$$

For V_1 , by (4.50), it is easy to check that

$$(4.68) \quad \mathbb{E}|V_1|^2 = \left(\mathbb{E} \int_{t_j}^T |\Phi_k(t_j) D_u \Psi_k(s)|^4 ds \mathbb{E} \int_{t_j}^T |f_x D_\theta x(s)|^4 ds \right)^{1/2} \leq K.$$

For V_2 ,

$$(4.69) \quad \begin{aligned} \mathbb{E}|V_2|^2 &\leq \mathbb{E} \left| \int_{t_j}^T \Phi_k(t_j) \Psi_k(s) D_u(f_x D_\theta x(s)) ds \right|^2 \\ &\leq K \mathbb{E} \left(\int_{t_j}^T |\Phi_k(t_j) \Psi_k(s)| \times (|D_\theta x(s) D_u x(s)| \right. \\ &\quad \left. + |D_\theta x(s) D_u Z^k(s)| + |D_u D_\theta x(s)|) ds \right)^2. \end{aligned}$$

We estimate the right side of (4.69) term by term. By Lemma 2.2 and Hölder's inequality,

$$(4.70) \quad \begin{aligned} &\mathbb{E} \left(\int_{t_j}^T |\Phi_k(t_j) \Psi_k(s)| |D_\theta x(s) D_u x(s)| ds \right)^2 \\ &\leq \mathbb{E} \left\{ \sup_{0 \leq s \leq T} |\Phi_k(t_j) \Psi_k(s)|^2 \left(\int_{t_j}^T (|D_\theta x(s)|^2 + |D_u x(s)|^2) ds \right)^2 \right\} \\ &\leq K \left(\mathbb{E} \sup_{0 \leq s \leq T} |\Phi_k(t_j) \Psi_k(s)|^{\frac{2p_0}{p_0-1}} \right)^{\frac{p_0-1}{p_0}} \left(\mathbb{E} \int_{t_j}^T (|D_\theta x(s)|^{2p_0} + |D_u x(s)|^{2p_0}) ds \right)^{\frac{1}{p_0}} \\ &\leq K < \infty. \end{aligned}$$

Similarly,

$$(4.71) \quad \begin{aligned} &\mathbb{E} \left(\int_{t_j}^T |\Phi_k(t_j) \Psi_k(s)| |D_u D_\theta x(s)| ds \right)^2 \\ &\leq \mathbb{E} \left\{ \sup_{0 \leq s \leq T} |\Phi_k(t_j) \Psi_k(s)|^2 \left(\int_{t_j}^T |D_u D_\theta x(s)| ds \right)^2 \right\} \\ &\leq \left(\mathbb{E} \sup_{0 \leq s \leq T} |\Phi_k(t_j) \Psi_k(s)|^{\frac{2p_0}{p_0-2}} \right)^{\frac{p_0-2}{p_0}} \left(\mathbb{E} \int_{t_j}^T |D_u D_\theta x(s)|^{p_0} ds \right)^{\frac{2}{p_0}} \\ &\leq K < \infty. \end{aligned}$$

Now, we estimate the left terms in the right side of (4.69). For any $k = 0, 1, \dots, N-1$, applying Lemma 2.1, one can get

$$\mathbb{E} \left(\int_{t_k}^T |D_\theta Z^k(s)|^2 ds \right)^{p_0} \leq K \left\{ \mathbb{E} |D_\theta Y^k(T)|^{2p_0} + \mathbb{E} \int_t^{t_{j+1}} |f_x D_\theta x(s)|^{2p_0} ds \right\} \leq K < \infty.$$

Therefore,

$$(4.72) \quad \begin{aligned} &\mathbb{E} \left(\int_{t_j}^T |\Phi_k(t_j) \Psi_k(s)| |D_\theta x(s) D_u Z^k(s)| ds \right)^2 \\ &\leq \mathbb{E} \left\{ \sup_{0 \leq s \leq T} |\Phi_k(t_j) \Psi_k(s)|^2 \left(\sup_{0 \leq s \leq T} |D_\theta x(s)|^4 + \left(\int_{t_j}^T |D_u Z^k(s)|^2 ds \right)^2 \right) \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \left(\mathbb{E} \sup_{0 \leq s \leq T} |\Phi_k(t_j) \Psi_k(s)|^{\frac{2p_0}{p_0-2}} \right)^{\frac{p_0-2}{p_0}} \\
&\quad \times \left\{ \left(\mathbb{E} \sup_{0 \leq s \leq T} |D_\theta x(s)|^{2p_0} \right)^{\frac{2}{p_0}} + \left[\mathbb{E} \left(\int_{t_j}^T |D_u Z^k(s)|^2 ds \right)^{p_0} \right]^{\frac{2}{p_0}} \right\} \\
&\leq K < \infty.
\end{aligned}$$

Hence, (4.67), together with (4.68)–(4.72), yields that

$$(4.73) \quad \mathbb{E}|I_{23}|^2 = \mathbb{E} \left| \int_{t_j}^t v_\theta(u) dW(u) \right|^2 \leq \mathbb{E} \int_{t_j}^t |v_\theta(u)|^2 du \leq K|t - t_j|.$$

Finally, by (4.60)–(4.66) and (4.73), one gets

$$\mathbb{E}|D_\theta Y_k(t) - D_\theta Y_k(t_j)|^2 \leq K|t - t_j|,$$

which deduces (4.59) by setting $\theta = t_j$. Now combining (4.57) with (4.58) and (4.59), we have the regularity of Z (4.44). \blacksquare

Remark 4.10. From the proof of Lemma 4.7 and Lemma 4.8, we can see that when $f = f(t, s, x, y)$ in BSDE (4.2), we only need $p_0 = 2$ in assumption (A4); but when $f = f(t, s, x, z)$, $p_0 > 2$ is needed.

4.2 Proof of Theorem 4.3

In this part, we prove our main result Theorem 4.3. Firstly, we need the following lemma on conditional expectation. One can refer to [15] for proof.

Lemma 4.11. For any $\varphi(\cdot) \in L^2_{\mathbb{F}}(\Omega \times (0, T); \mathbb{R}^n)$ and $0 \leq s < t \leq T$, write

$$\varphi_0 = \frac{1}{t-s} \mathbb{E} \left(\int_s^t \varphi(\tau) d\tau \middle| \mathcal{F}_s \right).$$

Then for any $\xi \in L^2_{\mathcal{F}_s}(\Omega; \mathbb{R}^n)$, it holds that

$$\mathbb{E} \int_s^t |\varphi(\tau) - \varphi_0|^2 d\tau \leq \mathbb{E} \int_s^t |\varphi(\tau) - \xi|^2 d\tau.$$

The following lemma is on the relation between $(Y^{\pi(\cdot)}(\cdot), Z^{\pi(\cdot)}(\cdot))$ and $(Y^{\pi(\cdot), \pi}(\cdot), Z^{\pi(\cdot), \pi}(\cdot))$.

Lemma 4.12. Let (A1)–(A4) hold. Then, for any $k = 0, 1, \dots, N-1$,

$$(4.74) \quad \sup_{k \leq j \leq N} \mathbb{E}|Y^k(t_j) - Y^{k, \pi}(t_j)|^2 + \mathbb{E} \int_{t_k}^T |Z^k(s) - Z^{k, \pi}(\tau(s))|^2 ds \leq K|\pi|,$$

where K is a constant depending only on L and T .

Proof. We split the proof into three steps.

Step 1. For any $k = 0, 1, \dots, N-1$ and $k \leq j \leq N-1$, denote

$$(4.75) \quad \begin{aligned} I_{k,j} &= \sup_{t_j \leq t \leq t_{j+1}} \mathbb{E}|Y^k(t) - Y^{k,\pi}(t)|^2 + \frac{1}{2} \mathbb{E} \int_{t_j}^{t_{j+1}} |Z^k(s) - \widehat{Z}^{k,\pi}(s)|^2 ds; \\ I_{k,N} &= \mathbb{E}|g(t_k, x(T)) - g(t_k, x^\pi(T))|^2. \end{aligned}$$

By Eq. (4.2) and (4.3), for $j \geq k$, we have

$$\begin{aligned} & (Y^k(t_j) - Y^{k,\pi}(t_j)) + \int_{t_j}^{t_{j+1}} (Z^k(s) - \widehat{Z}^{k,\pi}(s)) dW(s) \\ &= (Y^k(t_{j+1}) - Y^{k,\pi}(t_{j+1})) \\ &+ \int_{t_j}^{t_{j+1}} (f(t_k, s, x(s), Y^j(s), Z^k(s)) - f(t_k, t_j, x^\pi(t_j), Y^{j,\pi}(t_{j+1}), Z_0^{k,\pi}(t_j))) ds. \end{aligned}$$

Squaring and Taking expectation on both sides of the above equation, we obtain

$$(4.76) \quad \begin{aligned} & \mathbb{E}|Y^k(t_j) - Y^{k,\pi}(t_j)|^2 + \mathbb{E} \int_{t_j}^{t_{j+1}} |Z^k(s) - \widehat{Z}^{k,\pi}(s)|^2 ds \\ & \leq \left(1 + \frac{8\Delta_j}{\varepsilon}\right) \mathbb{E}|Y^k(t_{j+1}) - Y^{k,\pi}(t_{j+1})|^2 \\ & + \left(8 + \frac{\varepsilon}{\Delta_j}\right) L^2 \left\{ \mathbb{E} \left| \int_{t_j}^{t_{j+1}} \sqrt{s - t_j} ds \right|^2 + \mathbb{E} \left| \int_{t_j}^{t_{j+1}} x(s) - x(t_j) ds \right|^2 \right. \\ & + \mathbb{E} \left| \int_{t_j}^{t_{j+1}} x(t_j) - x^\pi(t_j) ds \right|^2 + \mathbb{E} \left| \int_{t_j}^{t_{j+1}} Y^j(s) - Y^j(t_{j+1}) ds \right|^2 \\ & + \mathbb{E} \left| \int_{t_j}^{t_{j+1}} Y^j(t_{j+1}) - Y^{j,\pi}(t_{j+1}) ds \right|^2 + \mathbb{E} \left| \int_{t_j}^{t_{j+1}} Z^k(s) - Z^k(t_j) ds \right|^2 \\ & + \mathbb{E} \left| \int_{t_j}^{t_{j+1}} \frac{1}{\Delta_j} \mathbb{E} \left(\int_{t_j}^{t_{j+1}} (Z^k(t_j) - Z^k(\tau)) d\tau \middle| \mathcal{F}_{t_j} \right) ds \right|^2 \\ & + \mathbb{E} \left| \int_{t_j}^{t_{j+1}} \frac{1}{\Delta_j} \mathbb{E} \left(\int_{t_j}^{t_{j+1}} (Z^k(\tau) - \widehat{Z}^{k,\pi}(\tau)) d\tau \middle| \mathcal{F}_{t_j} \right) ds \right|^2 \Big\} \\ & \leq \left(1 + \frac{8\Delta_j}{\varepsilon}\right) \mathbb{E}|Y^k(t_{j+1}) - Y^{k,\pi}(t_{j+1})|^2 \\ & + \left(8 + \frac{\varepsilon}{\Delta_j}\right) L^2 \left\{ K|\pi|^3 + |\pi|^2 \mathbb{E}|Y^j(t_{j+1}) - Y^{j,\pi}(t_{j+1})|^2 \right. \\ & \quad \left. + |\pi| \mathbb{E} \int_{t_j}^{t_{j+1}} |Z^k(s) - \widehat{Z}^{k,\pi}(s)|^2 ds \right\}. \end{aligned}$$

Now, choosing $\varepsilon = \frac{1}{2L^2}$, then for $|\pi| \leq \frac{1}{16L^2}$, one gets

$$(4.77) \quad \begin{aligned} & \mathbb{E}|Y^k(t_j) - Y^{k,\pi}(t_j)|^2 + \frac{1}{2} \mathbb{E} \int_{t_j}^{t_{j+1}} |Z^k(s) - \widehat{Z}^{k,\pi}(s)|^2 ds \\ & \leq (1 + 16L^2|\pi|) \mathbb{E}|Y^k(t_{j+1}) - Y^{k,\pi}(t_{j+1})|^2 \\ & + \left(8|\pi| + \frac{1}{2L^2}\right) L^2 \left\{ K|\pi|^2 + |\pi| \mathbb{E}|Y^j(t_{j+1}) - Y^{j,\pi}(t_{j+1})|^2 \right\}. \end{aligned}$$

In the above inequality, we use Lemma 2.2, Theorem 3.1 and Lemma 4.5. For simplicity, denote $b = 1 + 16L^2|\pi|$, $c = b|\pi|$. Then, by induction, we can get

$$(4.78) \quad \begin{aligned} I_{k,j} &\leq bI_{k,j+1} + cI_{j,j+1} + cK|\pi| \\ &\leq b^{N-j}I_{k,N} + \sum_{l=0}^{N-j-1} b^{N-j-l-1}c(b+c)^l I_{j+l,N} + cK|\pi| \sum_{l=0}^{N-j-1} (b+c)^l. \end{aligned}$$

For any k , by (A2) and Theorem 3.1,

$$(4.79) \quad I_{k,N} = \mathbb{E}|g(t_k, x(T)) - g(t_k, x^\pi(T))|^2 \leq K\mathbb{E}|x(T) - x^\pi(T)|^2 \leq K|\pi|.$$

Also, it is easy to check that,

$$(4.80) \quad \sum_{l=0}^{N-j-1} b^{N-j-l-1}c(b+c)^l = b^{N-j-1}c \sum_{l=0}^{N-j-1} \frac{(b+c)^l}{b^l} \leq b^N \left(1 + \frac{c}{b}\right)^N \leq e^{16L^2T} e^T,$$

and

$$(4.81) \quad cK|\pi| \sum_{l=0}^{N-j-1} (b+c)^l = cK|\pi| \frac{(b+c)^{N-j} - 1}{b+c-1} = Kb|\pi|^2 \frac{(b+b|\pi|)^{N-j} - 1}{b+b|\pi| - 1}.$$

Since

$$b + b|\pi| = 1 + (1 + 16L^2)|\pi| + 16L^2|\pi|^2 \leq 1 + (2 + 32L^2)|\pi|,$$

and $(b + b|\pi|) - 1 \geq 16L^2|\pi|$, (4.81) turns into

$$(4.82) \quad \begin{aligned} cK|\pi| \sum_{l=0}^{N-j-1} (b+c)^l &\leq K(1 + 16L^2|\pi|)|\pi|^2 \frac{(1 + (2 + 32L^2)|\pi|)^N}{16L^2|\pi|} \\ &\leq \frac{K}{16L^2} |\pi|(1 + 16L^2|\pi|)e^{(2+32L^2)T} \leq K|\pi|. \end{aligned}$$

Hence, (4.78), together with (4.79), (4.80) and (4.82), yields that

$$(4.83) \quad \mathbb{E}|Y^k(t_j) - Y^{k,\pi}(t_j)|^2 \leq K|\pi|.$$

That is the first part of (4.74).

Step 2. Now, we estimate $\mathbb{E} \int_{t_k}^T |Z^k(s) - Z^{k,\pi}(\tau(s))|^2 ds$. By (4.77), summing from $j = k$ to $N - 1$ leads to

$$\begin{aligned} &\sum_{j=k}^{N-1} \mathbb{E}|Y^k(t_j) - Y^{k,\pi}(t_j)|^2 + \frac{1}{2} \mathbb{E} \int_{t_k}^{t_N} |Z^k(s) - \widehat{Z}^{k,\pi}(s)|^2 ds \\ &\leq (1 + 16L^2|\pi|) \sum_{j=k}^{N-1} \mathbb{E}|Y^k(t_{j+1}) - Y^{k,\pi}(t_{j+1})|^2 \\ &\quad + (1 + 16L^2|\pi|)|\pi| \sum_{j=k}^{N-1} \{K|\pi| + \mathbb{E}|Y^j(t_{j+1}) - Y^{j,\pi}(t_{j+1})|^2\}. \end{aligned}$$

Hence, by (4.83),

$$\begin{aligned}
& \mathbb{E} \int_{t_k}^T |Z^k(s) - \widehat{Z}^{k,\pi}(s)|^2 ds \\
& \leq 32L^2 |\pi| \sum_{j=k}^{N-1} \mathbb{E} |Y^k(t_{j+1}) - Y^{k,\pi}(t_{j+1})|^2 + 2\mathbb{E} |Y^k(t_N) - Y^{k,\pi}(t_N)|^2 \\
& \quad - 2\mathbb{E} |Y^k(t_k) - Y^{k,\pi}(t_k)|^2 + (2 + 32L^2 |\pi|) |\pi| \sum_{j=k}^{N-1} \{K|\pi| + \mathbb{E} |Y^j(t_{j+1}) - Y^{j,\pi}(t_{j+1})|^2\} \\
& \leq K|\pi|.
\end{aligned} \tag{4.84}$$

Step 3. For any $k = 0, 1, \dots, N-1$, and $k \leq j \leq N-1$, denote

$$\bar{Z}^k(t_j) = \frac{1}{\Delta_j} \mathbb{E} \left(\int_{t_j}^{t_{j+1}} Z^k(s) ds | \mathcal{F}_{t_j} \right).$$

Then, by Lemma 4.7, Lemma 4.8 and Lemma 4.11 and (4.84), a direct calculation leads to

$$\begin{aligned}
& \mathbb{E} \int_{t_k}^T |Z^k(s) - Z^{k,\pi}(\tau(s))|^2 ds = \sum_{j=k}^{N-1} \mathbb{E} \int_{t_j}^{t_{j+1}} |Z^k(s) - Z^{k,\pi}(\tau(s))|^2 ds \\
& \leq 2 \sum_{j=k}^{N-1} \mathbb{E} \int_{t_j}^{t_{j+1}} (|Z^k(s) - \bar{Z}^k(t_j)|^2 + |\bar{Z}^k(t_j) - Z^{k,\pi}(t_j)|^2) ds \\
& \leq 2 \sum_{j=k}^{N-1} \mathbb{E} \int_{t_j}^{t_{j+1}} \left(|Z^k(s) - Z^k(t_j)|^2 + \left| \frac{1}{\Delta_j} \mathbb{E} \left(\int_{t_j}^{t_{j+1}} Z^k(u) - \widehat{Z}^{k,\pi}(u) du | \mathcal{F}_{t_j} \right) \right|^2 \right) ds \\
& \leq K|\pi| + 2 \sum_{j=k}^{N-1} \mathbb{E} \int_{t_j}^{t_{j+1}} |Z^k(u) - \widehat{Z}^{k,\pi}(u)|^2 du \\
& \leq K|\pi|.
\end{aligned}$$

That completes the proof. ■

Now, we are in the step to prove Theorem 4.3.

Proof of Theorem 4.3. By Lemma 4.12, we can see that $\sup_{0 \leq t \leq T} \mathbb{E} |Y(\tau(t)) - Y^{\pi(t),\pi}(\tau(t))|^2 \leq K|\pi|$ is true.

For the second term, $\mathbb{E} \int_0^T \int_t^T |Z(t, s) - Z^{\pi(t),\pi}(\tau(s))|^2 ds$, on the left side of (4.5), It is easy to check that

$$\begin{aligned}
& \mathbb{E} \int_0^T \int_t^T |Z(t, s) - Z^{\pi(t),\pi}(\tau(s))|^2 ds dt \\
& \leq \mathbb{E} \int_0^T \int_t^T |(Z(t, s) - Z(\tau(t), s)) + (Z(\tau(t), s) - \widehat{Z}^{\pi(t),\pi}(s)) \\
& \quad + (\widehat{Z}^{\pi(t),\pi}(s) - Z^{\pi(t),\pi}(\tau(s)))|^2 ds dt
\end{aligned} \tag{4.85}$$

$$\begin{aligned}
&\leq 3\mathbb{E} \int_0^T \int_t^T |Z(t, s) - Z(\tau(t), s)|^2 ds dt + 3\mathbb{E} \int_0^T \int_{\tau(t)}^T |Z(\tau(t), s) - \widehat{Z}^{\pi(t), \pi}(s)|^2 ds dt \\
&\quad + 3\mathbb{E} \int_0^T \int_{\tau(t)}^T |\widehat{Z}^{\pi(t), \pi}(s) - Z^{\pi(t), \pi}(\tau(s))|^2 ds dt.
\end{aligned}$$

By Lemma 2.4, one has

$$(4.86) \quad \mathbb{E} \int_0^T \int_t^T |Z(t, s) - Z(\tau(t), s)|^2 ds dt \leq C \int_0^T (t - \tau(t)) dt \leq C|\pi|.$$

For the third term on the right side of (4.85), by (4.84) and (4.2), one has

$$\begin{aligned}
&\mathbb{E} \int_0^T \int_{\tau(t)}^T |\widehat{Z}^{\pi(t), \pi}(s) - Z^{\pi(t), \pi}(\tau(s))|^2 ds dt \\
(4.87) \quad &\leq 2\mathbb{E} \int_0^T \int_{\tau(t)}^T |\widehat{Z}^{\pi(t), \pi}(s) - Z^{\pi(t)}(s)|^2 ds dt + 2\mathbb{E} \int_0^T \int_{\tau(t)}^T |Z^{\pi(t)}(s) - Z^{\pi(t), \pi}(\tau(s))|^2 ds dt \\
&\leq K|\pi|.
\end{aligned}$$

Now, we estimate $\mathbb{E} \int_0^T \int_{\tau(t)}^T |Z(\tau(t), s) - \widehat{Z}^{\pi(t), \pi}(s)|^2 ds dt$. By Eq. (1.1) and (4.3), for any $k = 0, 1, \dots, N-1$, one can easily calculate

$$\begin{aligned}
(4.88) \quad &\mathbb{E}|Y(t_k) - Y^{k, \pi}(t_k)|^2 + \mathbb{E} \int_{t_k}^T |Z(t_k, s) - \widehat{Z}^{k, \pi}(s)|^2 ds \\
&\leq \mathbb{E} \left| (g(t_k, x(T)) - g(t_k, x^\pi(T))) \right. \\
&\quad \left. + \sum_{l=k}^{N-1} \int_{t_l}^{t_{l+1}} (f(t_k, s, x(s), Y(s), Z(t_k, s)) - f(t_k, t_l, x^\pi(t_l), Y^{l, \pi}(t_{l+1}), \widehat{Z}_0^{k, \pi}(t_l))) ds \right|^2 \\
&\leq 2L^2 \mathbb{E}|x(T) - x^\pi(T)|^2 \\
&\quad + 2N|\pi| \sum_{l=k}^{N-1} \mathbb{E} \int_{t_l}^{t_{l+1}} |f(t_k, s, x(s), Y(s), Z(t_k, s)) - f(t_k, t_l, x^\pi(t_l), Y^{l, \pi}(t_{l+1}), \widehat{Z}_0^{k, \pi}(t_l))|^2 ds \\
&\leq K|\pi| + K \sum_{l=k}^{N-1} \mathbb{E} \int_{t_l}^{t_{l+1}} \left(|s - t_l| + |x(s) - x(t_l)|^2 + |x(t_l) - x^\pi(t_l)|^2 \right. \\
&\quad \left. + |Y(s) - Y^l(s)|^2 + |Y^l(s) - Y^l(t_{l+1})|^2 + |Y^l(t_{l+1}) - Y^{l, \pi}(t_{l+1})|^2 \right. \\
&\quad \left. + |Z(t_k, s) - Z^k(s)|^2 + |Z^k(s) - Z^{k, \pi}(t_l)|^2 \right) ds \\
&\leq K|\pi|.
\end{aligned}$$

Here, we use Theorem 4.2, Lemma 4.5, Lemma 4.12, Proposition 4.6 and (4.2). Now, (4.85), together with (4.86)–(4.88), yields that

$$(4.89) \quad \mathbb{E} \int_0^T \int_t^T |Z(t, s) - Z^{\pi(t), \pi}(\tau(s))|^2 ds dt \leq K|\pi|.$$

That completes the proof of the convergent speed of the Euler method for BSVIE (1.1). ■

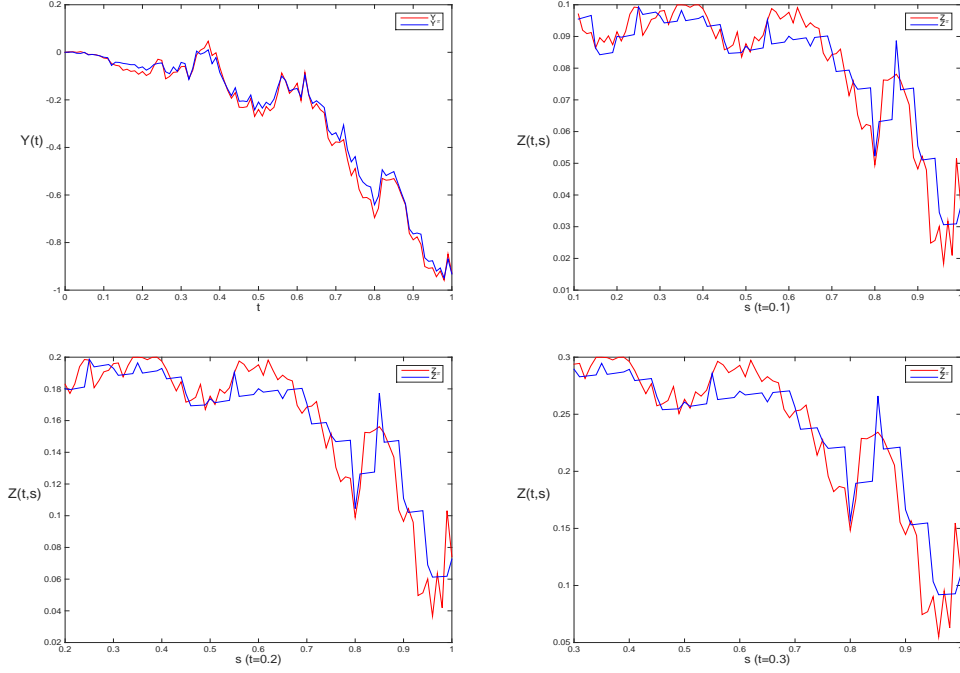


Figure 1: $(Y(t), Z(t, s))$ and its approximation $(Y^{\pi(t), \pi}(t), Z^{\pi(t), \pi}(\tau(s)))$. For $Z(t, s)$ and its approximation $Z^{\pi(t), \pi}(\tau(s))$, we choose $t = 0.1, 0.2, 0.3$ for one sample path $\omega \in \Omega$.

5 A Numerical example

In this section, we mainly present a numerical example. Consider the following BSVIE:

$$(5.1) \quad Y(t) = t \sin(W(1)) + \int_t^1 \frac{t}{2} \sin(W(s)) ds - \int_t^1 Z(t, s) dW(s), \quad t \in [0, 1],$$

with $T = d = n = 1$, which admits a unique solution $(t \sin(W(t)), t \cos(W(s)))$.

In Figure 1, choosing $N = 100$ (i.e. $|\pi| = 0.01$), we simulate true solution $(Y(t), Z(t, s))$ (in red) and its approximation $(Y^{\pi(\cdot), \pi}(\cdot), Z^{\pi(\cdot), \pi}(\cdot))$ (in blue). For the Z part, we take three cases: $t = 0.1, 0.2, 0.3$ and one sample path $\omega \in \Omega$.

Acknowledgement

This work was carried out during the stay of the author at University of Central Florida, USA. The author would like to thank the Department of Mathematics for its hospitality, and the financial support from China Scholarship Council. The author also gratefully acknowledges Professor Jiongmin Yong for stimulating discussions during this work.

References

- [1] C. BENDER AND R. DENK, *A forward scheme for backward SDEs*, Stochastic Process. Appl., 117 (2007), pp. 1793–1812.

- [2] C. BENDER AND S. POKALYUK, *Discretization of backward stochastic Volterra integral equations*, in Recent developments in computational finance, vol. 14 of Interdiscip. Math. Sci., World Sci. Publ., Hackensack, NJ, 2013, pp. 245–278.
- [3] M. A. BERGER AND V. J. MIZEL, *Volterra equations with Itô integrals. I, II*, J. Integral Equations, 2 (1980), pp. 187–245, 319–337.
- [4] B. BOUCHARD AND N. TOUZI, *Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations*, Stochastic Process. Appl., 111 (2004), pp. 175–206.
- [5] P. BRIAND AND C. LABART, *Simulation of BSDEs by Wiener chaos expansion*, Ann. Appl. Probab., 24 (2014), pp. 1129–1171.
- [6] S. CHEN AND J. YONG, *A linear quadratic optimal control problem for stochastic Volterra integral equations*, in Control theory and related topics, World Sci. Publ., Hackensack, NJ, 2007, pp. 44–66.
- [7] J. DOUGLAS, JR., J. MA, AND P. PROTTER, *Numerical methods for forward-backward stochastic differential equations*, Ann. Appl. Probab., 6 (1996), pp. 940–968.
- [8] N. EL KAROUI, S. PENG, AND M. C. QUENEZ, *Backward stochastic differential equations in finance*, Math. Finance, 7 (1997), pp. 1–71.
- [9] Y. HU, D. NUALART, AND X. SONG, *Malliavin calculus for backward stochastic differential equations and application to numerical solutions*, Ann. Appl. Probab., 21 (2011), pp. 2379–2423.
- [10] I. ITO, *On the existence and uniqueness of solutions of stochastic integral equations of the Volterra type*, Kodai Math. J., 2 (1979), pp. 158–170.
- [11] J. LIN, *Adapted solution of a backward stochastic nonlinear Volterra integral equation*, Stochastic Anal. Appl., 20 (2002), pp. 165–183.
- [12] J. MA, P. PROTTER, J. SAN MARTIN, AND S. TORRES, *Numerical method for backward stochastic differential equations*, Ann. Appl. Probab., 12 (2002), pp. 302–316.
- [13] G. N. MILSTEIN AND M. V. TRETYAKOV, *Numerical algorithms for forward-backward stochastic differential equations*, SIAM J. Sci. Comput., 28 (2006), pp. 561–582.
- [14] P. WANG AND X. ZHANG, *Numerical solutions of backward stochastic differential equations: a finite transposition method*, C. R. Math. Acad. Sci. Paris, 349 (2011), pp. 901–903.
- [15] Y. WANG, *A semidiscrete Galerkin scheme for backward stochastic parabolic differential equations*, Math. Control Relat. Fields, accepted.
- [16] J. YONG, *Continuous-time dynamic risk measures by backward stochastic Volterra integral equations*, Appl. Anal., 86 (2007), pp. 1429–1442.
- [17] J. YONG, *Well-posedness and regularity of backward stochastic Volterra integral equations*, Probab. Theory Related Fields, 142 (2008), pp. 21–77.

- [18] J. YONG, *Representation of adapted solutions to backward stochastic volterra integral wquations*, submitted.
- [19] J. ZHANG, *A numerical scheme for BSDEs*, Ann. Appl. Probab., 14 (2004), pp. 459–488.
- [20] X. ZHANG, *Euler schemes and large deviations for stochastic Volterra equations with singular kernels*, J. Differential Equations, 244 (2008), pp. 2226–2250.